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THE DESIGN OF REFLECTING PRISMS

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ABSTRACT. The displacement of a bundle of light resulting from two plane reflections is the equivalent of a "swing" of the emergent part of the bundle round the line of intersection of the two reflecting faces. Prisms can be designed by considering pairs of reflections. Graphical work for the directions of the axial ray and the disposition of the field round that ray can be worked out by means of a spherical diagram from which sufficient information can be derived to facilitate the construction of models (*a*) of the path of the axial ray and (*b*) of the full bundle of light.

From the second model (*b*) constructed reasonably to scale a wooden model of the prisms can be built up. The angles between faces of the prism can be computed accurately from the spherical diagram.

The method is particularly applicable to "skew" prisms in which three successive parts of the axial ray do not lie in a plane.

Binocular prism systems are dealt with in some detail.

INTRODUCTION

IN many optical arrangements it is necessary to provide, in addition to an axial system of lenses, a number of reflecting prisms for the purpose of shortening up the overall length of the instrument, or of erecting the image, or of putting a bend in the optical axis so that the direction of the emergent axial ray is different from that of the entrant ray. In some cases the prisms serve to effect a combination of these purposes.

When the changes in direction are rectangular, or all parallel to convenient planes, the problem of the most economical design of the prisms is not difficult, but cases may arise where more troublesome systems have to be dealt with and it is the main object of this paper to lay down a few general guiding principles upon which the design of such prisms should be based.

In all cases it is possible to separate the effect on the image formation that is due to the lens system from that which is due to the prisms, except as far as con-

cerns the aberrations; but dealing only with Gaussian optics the image positions and sizes can be determined by considering the coaxial lens system as one entity and the prism system as another. The final position, size, and manner of presentation of the image to the eye are found by compounding the two separate systems.

Some Elementary Principles

A plane reflection causes the image-forming light to proceed as if starting from a similar source on the opposite side of the reflecting surface. Suppose that a convergent pencil of rays (Fig. 1), all passing through a circular aperture *ABC* (*C* being the centre), would form an image *XYZ* (*Z* being the centre). This pencil is reflected by a plane and the image is formed not at *XYZ* but at *X'Y'Z'*, where the lines *XX'*, *YY'*, *ZZ'* (not shown in the diagram) are all at right angles to, and are bisected by, the plane of reflection. The image that is formed is similar to the image *XYZ* but there is this difference: it is of the opposite "hand." Looked at

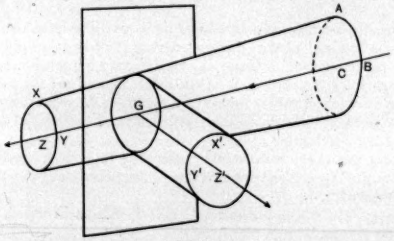

Fig. 1

from *G*, the point where the axial ray strikes the reflecting plane, the direction *X* to *Y* is an anti-clockwise direction round *Z*, whereas *X'* to *Y'* is a clockwise direction round *Z'*.

If, in place of a virtual optical image *XYZ* behind the mirror, we utilise a concrete object in front of the mirror, the difference is of the same character. Let the reader hold his right hand in front of a mirror; the image behind the mirror is a left hand. If the object is an engineer's screw with a right-hand thread upon it the image has a left-hand thread. If the object and the image could be both concrete bodies that could be picked up and handled independently of one another, by no possible means could the image be made to occupy the same space and position as the object. The left hand remains a left hand, and cannot possibly fill the space occupied by a right hand. The left-hand screw cannot possibly engage in a right-hand threaded nut.

An exactly similar effect is found with an optical image. Suppose there is in the field the image of a letter *R*. The single plane reflection will turn that into Я or ᴚ or ꓤ or any other position, depending on how the reflecting plane is placed

relatively to the image-forming light. All these letters are "looking-glass" letters; they differ from one another only in regard to rotation in the field round the optical axis. An axial lens system cannot produce any effect of this character. If the object is *R* the image is *R* or *Я* i.e. erect or inverted. The looking-glass image or "reverted" image, as it is sometimes called, is of quite a different character. A second reflection brings the image back to its original form. The right hand, twice reflected, becomes a right hand again.

It follows, then, that if the optical system is to produce an image that is erect, or inverted, or rotated, but not "reverted," there must be an even number of reflections. If a reverted image is required the number must be odd.

If we take the simplest case of two plane reflections, one important point emerges. Let the two planes, which must intersect in a line, be perpendicular to the plane of the paper (Fig. 2) and intersect in *H* so that *HA* and *HB* are plan sections of the two mirrors and *O*, also in the plane of the paper, is any point of

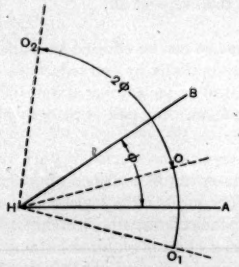

Fig. 2

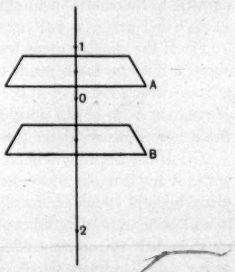

Fig. 3

the object. The point *O* is reflected in *HA* to O_1 where *HA* bisects OO_1 at right angles. The second reflection takes O_1 to O_2 where *HB* bisects O_1O_2 at right angles. It is quite evident that *HO* and HO_2 are equal and that the angle OHO_2 is twice the angle *AHB*. In other words, the determination of the point O_2 could be effected by swinging the line *HO* round *H* through twice the angle between the two mirrors. A similar statement applies to every point of the object, and as both object and image are of the same "hand," it follows that an image position after two plane reflections is uniquely found by a "swing" of this character. The direction of swing depends upon the order in which the reflections take place. If the first reflection had been in *HB* and the second in *HA* the swing would have to be in the opposite sense to that shown in the figure. There are thus two images produced as a result of reflections in two plane surfaces, but the two coalesce into one if the surfaces are at right angles, since each arrangement is equivalent to a swing of 180° round the line of intersection.

It should be noted that the displacement of the object to the position occupied by the image depends only on the inclination of the two mirrors and the position

of their line of intersection, and that there are, in consequence, any number of mirror arrangements that will produce the same effect, for all that is necessary is that they should intersect in the line H (Fig. 2) and be inclined to one another at the angle AHB.

A special modification of the general case of two plane reflections occurs when the reflecting surfaces are parallel. The line of intersection is at infinity and the angular value of the rotation is zero. But the two reflections are equivalent to a *translation* movement. This is illustrated in Fig. 3. The point o lying between the two parallel planes A and B is reflected first in A. The image is at the point 1. This image is now reflected in B and the image is at the point 2. The three points o, 1, and 2 clearly lie on a line perpendicular to the two planes and the separation between o and 2 is twice the distance between the two planes. If the order of reflection is reversed, i.e. B, A instead of A, B, the displacement is also reversed. The amount of the displacement is unaltered if the two planes A and B are moved parallel to themselves without alteration of their separation.

We thus arrive at two important results.

(i) If the displacement from object to image can be effected by pure rotation about a line the same result can be obtained optically by two reflections in plane mirrors which intersect in the line, and inclined to one another at half the amount of rotation from object position to image position. Any pair of planes which fulfil these conditions will have the same effect.

(ii) If the displacement from object to image can be effected by pure translation without any angular movement the same result can be obtained optically by two plane mirrors which are parallel to one another and at right angles to the direction in which the translation takes place. The perpendicular separation of the two mirrors must be half the amount of linear displacement that is required but any position of the pair of mirrors satisfying these conditions is effective in producing the same result.

Both a pure rotation and a pure translation can therefore be effected by a pair of reflections and there is in each case one degree of freedom (rotation in the first case and translation in the second) available in the disposition of the pair of mirrors.

The General Case of the Displacement of a Rigid Body

It is known that the displacement of a rigid body from any one position in space to any other is equivalent to a screw motion, that is to say, it can take place by sliding the body along a line by a certain amount and twisting it round the line through a certain angle.

Two reflections are therefore insufficient to cover the general case. If the reflecting faces are inclined to one another the displacement is purely one of rotation round the line of their intersection. If the faces are parallel the displacement is purely one of translation in a direction perpendicular to the two parallel planes. Four reflections are in general necessary to transfer a beam of light to any new position, but there are very many ways of doing this.

Fig. 4 illustrates in a diagrammatic manner one method of carrying out the

displacement by optical reflection. Let the line of the screw be PQ and the amount of the displacement $2d$. Let the angular rotation round PQ be 2ϕ. A and B are *any* pair of parallel reflectors each perpendicular to PQ and separated by a distance d. C and D are *any* pair of reflectors passing through PQ and inclined to one another at an angle ϕ. The order of the reflections is taken to be A, B, C, D. If A and B are interchanged the translation is of the same amount but in the opposite sense. If C and D are interchanged the rotation is of the same amount but in the opposite sense.

Now it is to be noted that B and C are two surfaces at right angles. In themselves they form a "roof" and we have already seen that with a roof the order of the two reflections is immaterial. Consequently, reflections in the order A, C, B, D will take the object to exactly the same image position as reflections in the order A, B, C, D. Again B and D are perpendicular and a further alteration in the order can be made, giving A, C, D, B as a third possible manner in which the reflections

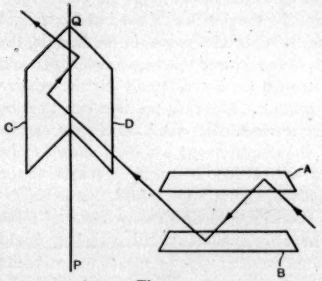

Fig. 4

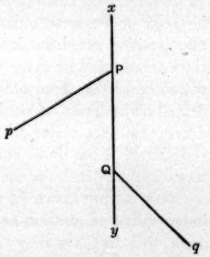

Fig. 5

may take place. Proceeding in this manner we find that there are in all six possible orders in which reflections can take place without affecting the final image position. These are:

A, B, C, D	A, C, B, D
A, C, D, B	C, A, B, D
C, A, D, B	C, D, A, B

In the first the translation A, B takes place first and the rotation C, D afterwards. In the last the rotation precedes the translation. In all the other four the first two reflectors form one roof and the second two form another. In each one of the four arrangements the roof edges both intersect the line PQ at points distant d from one another. In each case the two roof edges are inclined at an angle ϕ with one another.

Now it must be remembered that we started off with *any* pair of parallel planes A, B separated by d and perpendicular to PQ and with *any* pair of planes intersecting PQ and inclined to one another at an angle ϕ. Any arrangement of A, B

and any arrangement of C, D can be used as the starting-point for the purpose of rearranging the order of reflections and we get therefore a consideration of the problem of the four reflections that can be illustrated as in Fig. 5. PQ or xy is as before the axis of the screw. Imagine $pPQq$ to be a bent wire in which $PQ = d$ and pP and qQ are perpendicular to PQ and inclined to one another at the angle ϕ. That bent wire is free to move along the line xy and to rotate round the same line. A roof reflector is placed with its edge along pP and another with its edge along qQ. Either roof reflector can be turned round its own edge.

Any arrangement of the four reflecting faces that satisfies the above conditions will give the same displacement from object to image. Obviously one could go further since any two consecutive reflecting surfaces can be turned round their line of intersection provided the angle between them is unaltered.

It will be seen from the above that, although in the general case four reflections are needed, there is an enormous number of ways in which the four can be arranged to produce the required result.

It should be noted here that if we describe the position of the bundle of light in terms of the rectangular coordinates of three definite points in the bundle, the new position requires nine parameters to define it, but they are not independent since there must be three relationships between the coordinates to fix the relative distances of the three points from one another. The new position can thus be described in terms of six independent parameters. The same conclusion can be drawn from the screw motion to which the displacement has been shown to be equivalent. Four independent constants are needed to specify the axis of the screw and one each for the displacement along, and rotation round, that axis.

Now, to put down in general terms the equations of a plane demands three independent parameters and two planes have six available. At first sight it would seem, therefore, that any displacement could be dealt with by means of two plane reflections since there are just six independent parameters available to satisfy the six conditions of displacement. The geometry of Figs. 2 and 3 shows that such a conclusion is false. The reflecting planes must have ultimately one degree of freedom and the constants available for satisfaction of six equations are reduced to five. But although two reflections are insufficient the next possibility, viz. four reflecting surfaces, gives a great redundancy and the designer would have to ascribe arbitrary values to a considerable number of them if he were to attempt to design a prism in this manner.

What has been done above shows that there is a quadruply infinite number of ways of satisfying the conditions with two roof reflections. The general case probably gives a still greater degree of freedom.

Practical Aspects of Design

The above considerations show that the practical aspect of prism design demands that very severe restrictions be applied at the outset, and one of the most useful lies in working initially so as to obtain the angles of the prism system, leaving all considerations of linear dimensions to a later stage.

One can obtain an idea of the nature of the limitations that are thus introduced by considering the case of a double reflection prism which displaces the bundle of light into some definite position. If the prism is moved about bodily, but without alteration of the *direction* which is occupied by the line of intersection of the two reflecting faces, the image bundle will move about also, but without any form of rotation. If the axial ray is vertical it will not depart from the vertical; neither will the disposition of the focal plane round the axial ray be altered. Movement of the prism in this manner can displace the image position into any position that involves translation only, except that it cannot effect a translation parallel to the line of intersection of the two reflecting surfaces. Consequently if the designer produces a prism that will twist the bundle of light into the right directions he is still able to cope with two translational degrees of freedom but fails with the third if he wishes to limit himself to two reflections only.

In many cases the design of the particular instrument is such that the inability to satisfy all the rigid mathematical requirements is of small importance. It only means in most cases that the ocular must be moved to a slightly different position. If that position is hopelessly impracticable the two reflection systems must be abandoned and four reflections adopted.

Spherical Representation

As long as we are dealing only with the *directions* which the axial ray and the configuration of the focal plane occupy in space, it is much more convenient to work with a diagrammatic representation of them upon a spherical surface.

In Fig. 6 let a point A be chosen upon the surface of a sphere such that the outward radius to A is parallel to and in the same sense as the axial ray before entering the prism system. Let X represent the axial ray in the image space. The focal plane in the object space might conveniently be described in terms of a clock-face round the axis as centre. The "12 o'clock" and "3 o'clock" directions can be represented upon the sphere by short lines AB and AC, and in the image space by XY and XZ.

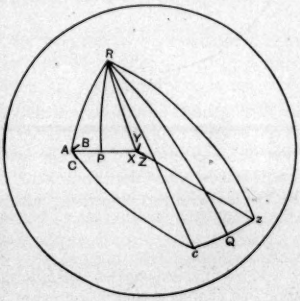

Fig. 6

Now AB, looked on as a part of a great circle, has two poles and c is that one shown in the figure. The point z is the corresponding pole of the great circle of which XY forms part. If on the sphere A is moved to X and c to z then the required displacement will have been made and the clock-face round A transferred to the clock-face round X. Draw the great circle AX and bisect it, drawing PR a

great circle at right angles to AX. Similarly bisect the great circle cz by another great circle QR at right angles to cz. PR and QR meet in R.

Now $AR = XR$ and $cR = zR$.

Hence if we think of R as the pole of the earth, A and X lie in the same parallel of latitude and c and z lie on another parallel of latitude. In the two triangles ARc and XRz, $AR = XR$, $cR = zR$, and $Ac = Xz = 90°$. The two triangles are equal and ARc can be displaced to XRz.

The displacement of the figure ABC upon the surface of the sphere to a position XYZ is thus accomplished by a rotation round the radius to the point R, the necessary amount of rotation being the angle ARX. If the bundle of light be allowed successively to strike two mirrors inclined at an angle $\frac{1}{2}$ ARX with one another and having their line of intersection parallel to the radius drawn to the point R, these two mirrors will have accomplished what is wanted, namely, to present the focal plane in a certain direction and disposed in the requisite manner round the axial ray.

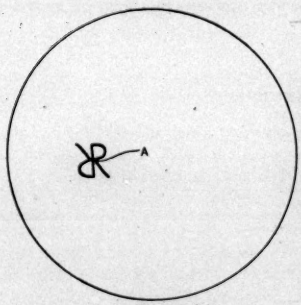

Fig. 7

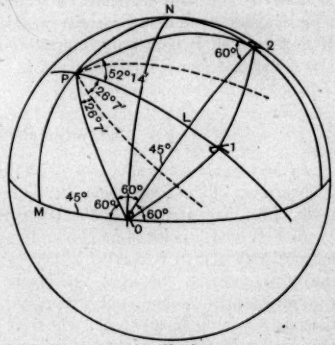

Fig. 8

The general conclusion can thus be reached that provided the image position is not of great importance two reflections are sufficient to effect any required alteration in the direction of the axial ray and orientation of the field round that ray, but it is to be noted that, even under such restrictions, there are cases which cannot be worked with two reflections, but require four.

The Prism Binocular

A notable instance of the failure of the two surfaces to effect the required direction change is that of the prism binocular. Here the axial ray must come out parallel to its original direction and the field be rotated 180° round the axial ray.

In place of the clock-face representation of the focal plane, let it be supposed that the objective is looking at a distant inverted letter R so that the field without

the prism contains an erect *R*. The prism system must present the field to the ocular in such a manner that the axial ray is parallel to its original direction and the field is rotated round that axis inverting the letter *R* again. On the sphere the points representing the direction of the unreflected and reflected rays (*A* and *X* of Fig. 6) are now coincident and Fig. 7 shows the corresponding spherical diagram, the middle of the upstroke of the *R* being taken to be the axial direction. The point *A* is thus the point round which rotation must take place and the amount 180° requires a roof prism *with the edge along the axis* of the objective. That is to say, the light is incident on each surface at grazing incidence and the prism binocular cannot be designed with two reflecting surfaces; four at least are necessary.

It must at the same time be remembered that a Dove roof prism would change that grazing incidence to something at a finite angle with the surfaces, but if placed inside the binocular it produces very bad aberration and if placed outside it seriously limits the field of view.

Practically the prism binocular can only be built with two reflections when there is a reasonably large alteration in the direction of the axial ray.

Another Example

A case in which two reflections are sufficient to produce the required alteration in the position of the image is the following. An objective has its axis at right angles to the plane of the paper and produces an image in the focal plane which is examined by means of an eyepiece. For purposes of explanation it is assumed that the object of which the objective produces an image is a distant inverted letter *R* so that the field contains an erect *R* with the upstroke in the middle vertical line as shown in Fig. 8. The optical designer finds it more convenient to put his eyepiece in a different position and needs a prism with two reflecting surfaces to bring about the necessary transference.

The manner in which the direction of the image is to be presented can be envisaged by supposing that the field is formed on the surface of a globe marked out in meridians and parallels. The first image, R_0, is at the equator in longitude 0° and the upstroke of the *R* lies along the meridian. The point 2 (R_2) representing the axial ray as presented to the ocular is in latitude 60° N., longitude 90° E. and the upstroke of the *R* lies along the meridian of the new position. The problem is then to find the radius of the sphere round which the letter R_0 can be swung round to position R_2. If we join the axial rays by a great circle the length of the arc is clearly 90°. Further this great circle makes an angle of 60° with the equator at R_0. The bisector *PL* at right angles to R_0R_2 must contain the axis of swing.

The upstroke of the R_0 is the polar of the point in latitude 0°, longitude 90° W. and the upstroke of the R_2 is the polar of the point R_0. Hence in swinging R_0 to R_2 these two poles of the upstrokes must also be transferred, which can be done by bisecting the piece of the equator between 90° W. and R_0 by a great circle at right angles to it, i.e. by the meridian 45° W. This also contains the axis of swing, and the point *P* round which the image must be swung is found by the intersection of

the great circle PL with the meridian 45° W. The angle of swing is R_0PR_2 and therefore the two reflecting surfaces must have their reflecting planes intersecting along the radius to P, and the supplement of the angle between their faces must be $\frac{1}{2}R_0PR_2$, i.e. equal to angle R_0PL.

The geometry of the figure shows that since R_0L and R_0M are each 45° and the angles at M and L both right angles, PR_0M and PR_0L must each be 60°. If x is the latitude of P and ϕ the supplement of the angle between the two reflecting surfaces, we obtain

$$\sin 45° = \tan x \tan 30°$$

or
$$\tan x = \sqrt{1{\cdot}5} \qquad \text{giving } x = 50° 27'$$
$$\cos \phi = \cos 45° \cos 30°$$
$$= \sqrt{0{\cdot}375} \qquad \text{or } \phi = 52° 14'$$

and
$$\sin 30° = \tan 45° \cot PR_0$$

or
$$\tan PR_0 = 2 \qquad \text{or } PR_0 = 63° 26'.$$

The edge of the prism, i.e. the line of intersection of the two reflecting faces, must thus be inclined at an angle of 63° 26′ to both the old and new directions of the axial ray. The two faces of the prism must be inclined at an angle of 127° 46′.

The design of the prism is not yet complete since the position of the two reflecting faces is not determined. Suppose they are arranged so that in their spherical representation one bisects the angle LPR_0 and the other the angle LPR_2. The two surfaces are shown in pecked lines. The intermediate, reverted, image will then be in the great circle through PL at a distance $PR_1 = PR_0 = PR_2 = 63° 26'$ and a further solution of spherical triangles shows that the axial ray will make an angle of 23° 11′ with each reflecting face.

If the swing had been in the opposite sense round P the inclination would have been the smaller angle 52° 14′ and the order of reflection would have been reversed, and there would have been the additional difference that the reflecting surfaces would be the external bisectors of the angles LPR_0, LPR_2.

General Consideration of the Binocular Prism System

We have seen that the inversion of the image for a prism binocular cannot be effected by two reflections. It is known that there are many ways in which the inversion can be effected by four, and these will be considered in some detail though obviously the methods are so multiply infinite that it is impossible to do more than treat of different types.

Dealing only with the directional representation of the field in a spherical diagram, the point 0 (Fig. 9) represents the initial axial ray and 2 the direction after two reflections. After two more reflections the axial ray must return to 0. The optical reflections must thus be equivalent to a swing round some point P to take the axial ray from 0 to 2, and then round some other point Q to take the axial ray back to 0. At the same time the field is to be rotated by 180° so that at 0, R represents the initial, and $\mathcal{R}$ the final field. At 2 the field is $\mathcal{R}$.

Both the points P and Q lie in the great circle which bisects o2 at right angles. To obtain the points P and Q we could utilise the poles of the upstrokes as in the previous example but it is convenient to notice from Figs. 6 and 8 that the field can be swung from the first position to that occupied after two reflections just as if it were firmly and rigidly attached by wires to the axis of swing. Thus in Fig. 9 the upstroke of the R_0 making an angle θ with Po swings round to the R_2 position and still makes an angle θ with $P2$. The angles which the upstroke makes with $Q2$ and Qo for the second swing are also equal to one another (ψ). The diagram $oP2Q$ has symmetry with respect to the great circle PQ and it follows immediately that

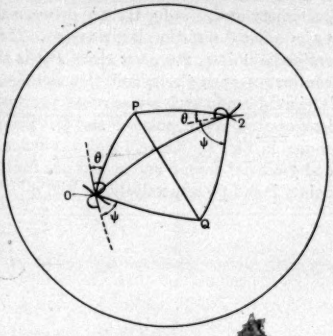

Fig. 9

$$\theta + \psi = 180° - \theta - \psi$$

or $$\theta + \psi = 90°.$$

It follows from this that, if we do not specify the position of the point 2, any two points P and Q upon the sphere will be suitable axes of swing provided the angle $PoQ = 90°$ and provided the angle or the supplement of the angle (according to the nature of the swing) between one pair of reflecting surfaces is oPQ and between the other pair is oQP.

In considering the most general case of the binocular prism system there is no loss of generality in treating the point o as fixed upon the sphere; it is merely a case of holding the binocular with its axis in the direction specified by o. Neither is there loss in generality in treating oP and oQ as fixed perpendicular great circles upon the sphere; to do so only involves the condition that the binocular must be orientated in some specified manner round its axis. With these conditions all possible cases are considered if we investigate the prism systems corresponding to all possible values of the arcs oP and oQ.

The complete investigation of all practicable values of these arcs in steps, say of 5°, and within reasonable limits, might perhaps be undertaken by the British Scientific Instrument Research Association and there would then be available for designers a full examination of all possible forms of the binocular prism system that involve four reflections. It is impossible to say whether among the many systems that would thus be considered there might be any which would be of practical value from the point of manufacture, but if only one system were found which gave a prism that could be manufactured from a single piece of glass without re-entrant angles, the work entailed in the general examination would be well worth while.

Existing Cases of the Binocular System

When we come to actual practical examples of prism systems that are in use, it appears that the general case has by no means been completely explored. The customary design is for the ray between the two prisms, i.e. ray 2, to be parallel to its original direction but reversed. The points 0 and 2 on the sphere are thus antipodal points; the great circle PQ is their equator and consequently $oP2$ is a continuous great circle, and $oQ2$ another great circle at right angles to the first. The axial ray in such a case meets every reflecting surface at 45° and the reverted images occur at the points P and Q. The image positions are then 0, 1, 2, 3, 4 as shown in Fig. 10, where 4 and 0 coincide, 2 being their antipodes. The points 1 and 3 are 90° from 0 and 2, and 90° from one another, and are identical with the points P and Q, respectively, of Fig. 9.

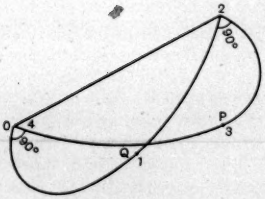

Fig. 10

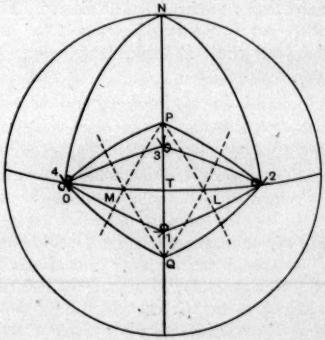

Fig. 11

The same lune-shaped diagram also exists in which 1 represents the initial and final rays and the sequence of image positions is in the same consecutive order round the line.

Another large sub-division of the general spherical diagram is that in which Q (Fig. 9) lies in the great circle 02. P must then be the pole of this great circle and the angular swing round P is equal to the arc 02. The swing back round Q is 180°, the reflecting surfaces forming a roof with its edge along Q. An illustration of the prism system for the case of $02 = 120°$ is the Sprenger prism and it has the merit that the axial ray is inclined at moderate angles with the four reflecting surfaces. These angles are 30°, 30°, 37° 46′, 37° 46′. Another point in its favour is that it can be made from a single block of glass.

A group of prism systems which have not apparently been used in the binocular is that in which P and Q (Fig. 9) are equidistant from the great circle 02, and the arc 02 is less than 180°. The general case of this group is shown in Fig. 11. The

first swing is round the point P through the angle $oP2$. Consequently, the supplement of the angle ϕ between the two reflecting faces is the angle oPQ. Symmetry will be preserved and the axial ray will meet all four reflecting planes at the same angle if the reflecting surfaces bisect the angles oPQ, $2PQ$, oQP, $2QP$.

The image of R in the first surface is a reverted R at the point 1 in PQ, where $P1 = Po$. The second reflection moves this reverted R to R_2, where the upstroke is now on the great circle 02. The third image is a reverted R again in the great circle PQ but above the arc 02 and the final image is back at o again, but inverted. Let PQ and 02 intersect at T. Let $PT = x$, $Po = y$, and $oT = \theta$. The angles PoQ and $P2Q$ being right angles, the angles PoT, etc. must each be 45° and from the spherical triangle oPT we have

$$\cos\phi = \sqrt{0{\cdot}5}\cos\theta$$
$$\tan x = \sin\theta$$
$$\tan y = \sqrt{2}\tan\theta.$$

Let p be the inclination of the axial ray to the first reflecting surface. The arc 01 is $2p$ and we obtain

$$\begin{aligned}\cos 2p &= \cos^2 y + \sin^2 y\cos\phi\\ &= \cos^2 y\,(1 + \tan^2 y\cos\phi)\\ &= (1 + \sqrt{2}\sin\theta\tan\theta)/(1 + 2\tan^2\theta).\end{aligned}$$

The values of ϕ, x, y, etc. for different values of θ between 60° and 90° are given in the accompanying table.

θ	ϕ	x	y	p	$T1$
°	° ′	° ′	° ′	° ′	° ′
60	69 18	40 54	67 48	31 45	26 54
65	72 37	42 11	71 45	34 13	29 34
70	76 0	43 13	75 34	36 36	32 21
75	79 27	44 1	79 16	38 54	35 15
80	82 57	44 34	82 54	41 5	38 20
85	86 28	44 53	86 28	43 8	41 35
90	90 0	45 0	90 0	45 0	45 0

In all these cases (except when $\theta = 90°$) the prisms are of the skew variety, that is to say, the three parts of the axial ray involved in two consecutive reflections do not lie in a plane.

Construction of a Model of the Axial Ray

It is useful at this stage to show how a model of the various directions of the axial ray can be constructed. Take the case of $\theta = 60°$ in the table in the previous paragraph. If in Fig. 11 the great circle $oT2$ is supposed to be horizontal, the initial ray o may be taken to be a horizontal line in the vertical plane No. The next ray 1 after the first reflection lies in the vertical plane $N1$, i.e. at an angle of 60° (or rather 120°) with the first, but the ray is no longer horizontal. It is at an angle $T1$

(26° 54′) below the horizontal. The third part of the axial ray is again horizontal but in a vertical plane that has again been turned through an angle of 60° from the previous one, and so on.

Now construct a cardboard model as shown in Fig. 12. Draw six vertical lines *A*, *B*, *C*, *D*, *E*, *F*. Across the first space draw a horizontal line 0. From the point where 0 meets *B*, carry on the line 1 downwards at an angle of 26° 54′ below the horizontal. From the point where this line meets *C* continue 2 in a horizontal direction. In the space between *D* and *E*, the line 3 is inclined at 26° 54′ upwards and in the last panel it proceeds horizontally. Now take this cardboard and bend

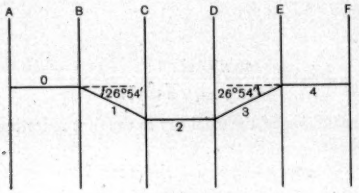

Fig. 12

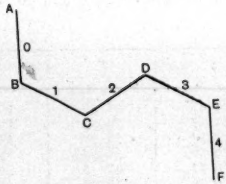

Fig. 13

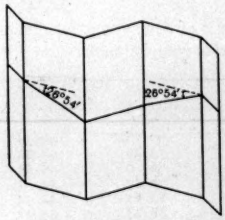

Fig. 14

it along the lines *B*, *C*, *D*, and *E*, making an angle of 120° in each case but arranged so that the cross-section is that shown in Fig. 13. The vertical planes standing on this plane are represented in this sphere (Fig. 11) by *No*, N_1, N_2, N_3, and N_4. If the card is correctly bent up to the proper angles, the line 01234 takes up the position in space occupied by the axial ray in travelling through the prism system (Fig. 14).

Instead of drawing the diagram on cardboard, a sounder method is to construct the vertical planes solidly in wood and to draw the various parts of the axial ray to lie on the several faces at the proper angles with the edges. Cuts can then be incised in the wood along the axial ray and a wire bent so as to lie along the cuts so made. In this way we can obtain a model in wire of the axial ray, the accuracy of direction, etc. depending upon the care with which the model is made. The

next stage is to affix at each elbow of the wire a small plane disc to represent the reflecting surface. The disc can be soldered to the wire but care must be taken that the disc is perpendicular to the plane containing the two parts of the wire at the elbow and is equally inclined to those two parts.

The wooden block, the wire model of the axial ray and a second model with the discs soldered to it are shown in the photograph (Fig. 15). Models of the full bundle of light and the completed prism are also shown in the photograph.

If care is taken to get the wire model reasonably accurate a fair notion may be obtained at this stage of what the prism will look like.

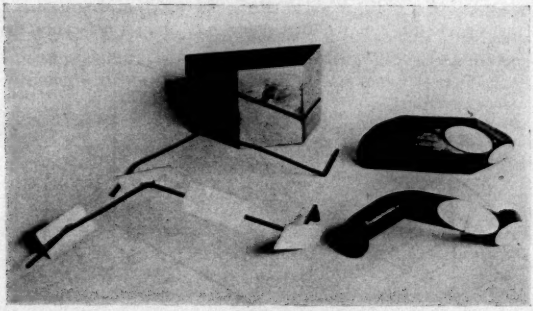

Fig. 15

The Binocular Prisms with Six Reflections

Very few attempts seem to have been made to design a binocular prism system in which there are six reflections. The project does not sound very practical, but the number of ways in which it could be attempted is so great that there may possibly be among the number prisms that are not of too difficult construction, and in which the length of glass is not unduly great.

A series of six reflections involves swings round three separate points, and three separate positions of the letter R upon the sphere. No general investigations will be made, but one case considered in which the three positions are equidistant from one another. In Fig. 16 the points 0, 2, 4, 6 represent the directions respectively. The points 0 and 6 coincide. The triangle is thus equilateral, all sides being equal and all angles equal. If the sides are known, the angles can be computed and vice versa. Since the triangle is equilateral, the bisectors of the angles of the triangle are also at right angles to the opposite sides, and meet at angles of 120° with one another in a point Z. The arcs $Z0$, $Z2$, $Z4$ are also obviously equal to one another. The centre round which the R swings from 0 to 2 is necessarily some point in the great circle

through 4 and Z, say the point C. Suppose the field of view to be represented by a clock-face, and let the 12 o'clock of this clock-face point originally (i.e. at the point o) towards Z. The swing takes place just as if the clock-face were rigidly attached to C and swung round C. Let the angle CoZ be θ. Then when the clock-face reaches Z, 12 o'clock will no longer point to Z but 2θ away from it. Relatively to the great circle joining the centre of the clock-face to Z, the clock may be said to have rotated anti-clockwise through 2θ.

The second swing from 2 to 4 must take place round some point A in oZ, and if we have exactly the same kind of swing as before, that is to say, if we make $ZA = ZC$, the twelve o'clock point will turn round another 2θ by the time it reaches the point 4. A similar swing round B ($ZB = ZA = ZC$) gives a further rotation of 2θ so that we find that after the three swings, i.e. after six reflections, the clock-face will have turned through 6θ. What is required is that after the six reflections 6 o'clock of the clock-face must point to Z. Hence 6θ must equal 180° or 540° or 900° or any multiple of 180°. This gives $\theta = 30°, 90°, 150°, 210°$, etc.

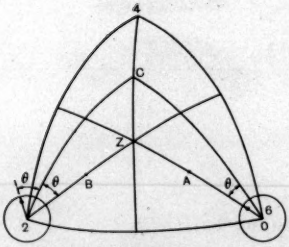

Fig. 16

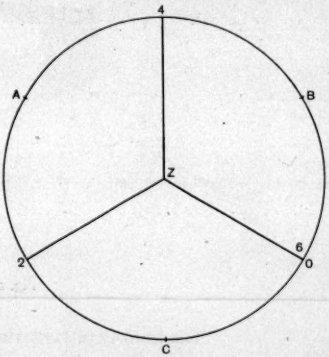

Fig. 17

In point of fact there are really only two separate cases. The spherical figures for 30°, 150°, 210°, etc. are essentially the same save that the series of swing positions will in some cases be 0246 and in others 0426. The 90°, 270°, etc. values are also all of the same character, and we thus get two separate arrangements according to whether θ is 30° or 90°.

Take the latter case first. We have still to assign lengths to the arcs 02, 24, etc. One simple case is that in which each is 120° and the three points 0, 2, 4 then lie on a great circle of which Z is the pole (Fig. 17). The swing points A, B, C lie also on this great circle. Now the swing that takes o to Z round C is 180° and the two reflecting surfaces that will do it form a roof with the edges in the direction of the point C.

If 12 o'clock points upwards from the plane of the paper at *C* it will point downwards at 2, upwards at 4, and downwards again at 6. We thus get the inversion of the field that is required. Fig. 18 shows the plan view of the prism, which is an equilateral triangle with a roof along each edge. It has no practical value since the light cannot get into it; or, if in, could not get out. But modified into the form shown in Fig. 19, in which the prism is made in two parts and cemented, it could be made to work.

Construction of Models showing the Full Pencil of Light

The wire models illustrated in Fig. 15 show only the disposition of the axial ray, and not the full pencil of light that has to pass through the prism. Further, in constructing the models no regard was paid to the lengths of the parts of the axial ray that lie between one reflection and the next. In practice, the assignment of suitable lengths to these parts is of the highest importance. Each reflecting

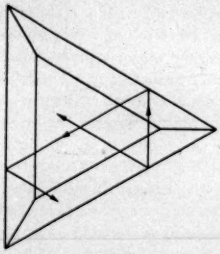

Fig. 18

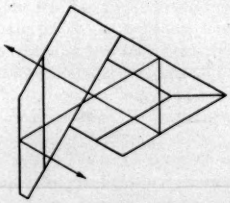

Fig. 19

plane cuts the pencil of light in an elliptical section. Two consecutive reflecting faces intersect in a straight line. In Fig. 20 *PQ* is the line of intersection of the two consecutive reflecting faces *EPQ* and *FPQ*, the axial ray striking at *A* and *B* respectively.

If the pencil of light is cylindrical, i.e. the objective and the field are the same size, then the light incident upon the first surface lies inside an ellipse of which *A* is the centre and on *B* inside another ellipse of which *B* is the centre. The minor axes of these two ellipses are each equal to the diameters of the cross-section of the bundle of light, the major axis being greater than that diameter in the ratio of sec *i* : 1, where *i* is the inclination of the axial ray to the normal.

If the field is bigger or smaller than the aperture, the ellipses will vary in respect of their minor axes as well as their major axes, and the axial ray will no longer strike at the centre of the ellipse.

In any case correct design demands that neither in the reflecting face *EPQ* nor in the face *FPQ* should either ellipse be cut by the line *PQ*. Any attempt at computing the position of the faces of the prism so that they are large enough to

accommodate the full bundle of light is hopeless, except in very simple cases. It is much easier to make an actual model reasonably to scale in wood, of the bundle of light as it passes through the prism.

For simplicity we will, for the present, deal with a cylindrical bundle. In Fig. 21 this cylinder is shown falling obliquely upon the horizontal plane representing the reflecting surface. The light, after reflection, fills the cylinder shown in

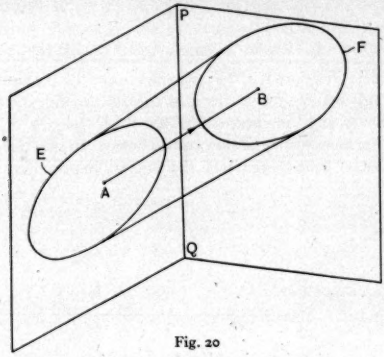

Fig. 20

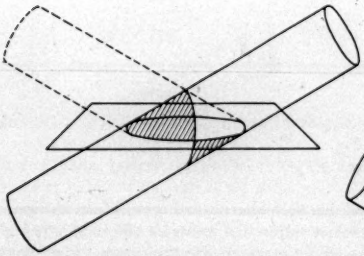

Fig. 21

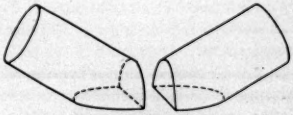

Fig. 22

pecked lines, and these cylinders both stand upon the same elliptical area of the reflecting face. The angle between the axial ray and the reflecting surface is known from the spherical figure. If we cut the cylinder through at the correct angle with the axis, we shall find the exact size and shape of the ellipse.

After cutting the cylinder, the lower portion below the mirror can be turned into the position of the bundle after reflection. The toes of the two parts are cut away (Fig. 21, shaded parts) and the two parts butted and stuck together, as shown in Fig. 22. This gives us an actual model of the reflection.

The next reflection requires a second section of the cylinder, and the position of this section must be such that it does not cut the ellipse which forms the base on which Fig. 22 stands.

As long as all the sections of the cylinder are such that the major axes all lie in one meridian plane, and the minor axes in another, the construction is simple, but in the skew prisms that we have been considering this is not the case and great care is needed in order that the planes of section may be correctly orientated round the axis of the cylinder.

The spherical geometry of two plane reflections can be seen from Fig. 11. o is the direction of the incident ray, and o1 the plane of the first reflection. The direction of the reflected ray 1 is formed by drawing oM1, a great circle, at right angles to PM, the plane of the mirror, and of such length that oM = M1. The arc oM represents the inclination of the axial ray to the plane of reflection, and represents therefore the angle to the axis of the cylinder at which it must be cut. The plane of section is an ellipse of which the major axis lies in the plane of the reflection, i.e. the plane oM1. The major axis is therefore the line represented by M, and the minor axis is in the great circle PM but 90° away from M. It is the pole of oM1 and is at right angles both to the incident and to the reflected rays.

After reflection at the second surface PL, in which the graphical construction is similar, we get for the direction of the axial ray after two reflections the point 2. The plane of the first reflection, i.e. the plane containing incident and reflected rays, is oM1, of the second 1L2. The angle 012 can be computed by spherical trigonometry, and from it can be determined the orientation of the end of the major axis of the ellipse with respect to the surface of the cylinder.

To make moderately accurate models, the cylinder must be cut with some care. In making those illustrating this paper the cylinder was held in a chuck mounted upon the slide rest of a lathe. A circular saw was mounted between the lathe centres. The chuck on the slide rest could be turned round a vertical axis by means of an angle scale. It could also be turned round its own axis by another scale. The first of these scales is set to the angles with the axis of the cylinder at which the section is to be cut. The second is rotated for each section in order that it may be cut with the plane containing the major axis at the correct orientation. The amount of rotation is complicated by the fact that after making each section the part sawn off after having its toe cut off has to be turned round 180° before being stuck on to the part remaining in the chuck. With the wire model to help, and with the angles computed from the spherical diagram, it is not difficult, after a few mistakes, to evolve a working rule for the amount and the direction of each rotation of the chuck round its axis in order to present the standing part correctly for the next section; but I hesitate to put down a working rule here since with a different arrangement for holding the work a different rule would be necessary.

When the work is correctly presented to the saw it is easy to traverse the slide rest in such a manner that each section just misses the previous one and so gives the most economical manner of designing the prism.

Cases, however, may arise where this economy is undesirable. When the model

of the bundle of light has been cut up and stuck together again, it may be found that the plane faces represented by the reflection ellipses involve re-entrant angles in the block of glass, and that some of them can be avoided by allowing greater lengths in some parts of the axial ray. The particular model in question will always be a guide here, and, if the re-entrant angles cannot possibly be avoided, the prism must be built up of parts instead of being made from a solid block.

When the bundle model has been finally made, it will indicate immediately what parts of the various flats upon the surface are necessary for proper reflection and what parts can be cut away for lightness, or possibly, to provide means for holding the prism in its seating.

Conical Bundles

Where the field and aperture are not of the same size, the bundle is conical and not cylindrical. In cutting the toes off the two parts they must not be sawn through the centre of the ellipse but through a point easily determinable if a meridional section of the bundle perpendicular to the reflecting surface is drawn out to scale.

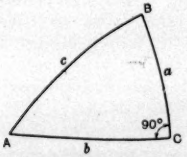

Fig. 23

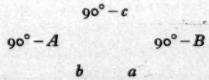

Fig. 24

APPENDIX

Spherical Trigonometry

Many people fight shy of spherical trigonometry on account of the complicated formulae. In point of fact almost all the spherical trigonometry necessary in prism design is concerned with the solution of right-angled triangles, and for them the whole of the formulae can be expressed in a single rule. In Fig. 23 the triangle *ABC* has a right angle at *C*, *A* and *B* are the two remaining angles, and *a*, *b* and *c* are the angular values of the sides, *c* being the hypotenuse. To begin with, write down, in order, the five parts of the triangle, omitting the right angle, and putting them in a pentagonal form, as shown in Fig. 24. The two sides *a*, *b* containing the right angle go down as they stand; each of the others, viz. *A*, *c*, and *B*, must be subtracted from 90°. The rule is as follows: the sine of any one of the five parts

is equal to the product of the tangents of the two parts adjacent to it, or to the product of the cosines of the two parts opposite to it. Thus

$$\sin b = \cot A \tan a = \sin c \sin B$$

$$\sin (90° - c) = \cos a \cos b, \text{ etc.}$$

It will be found that the mastery of this one rule for right-angled triangles makes all the trigonometry connected with spherical diagrams comparatively simple.

DISCUSSION

Professor L. N. G. Filon: I would like to congratulate Capt. Baker on the skill with which he has succeeded in making appear easy calculations that are apt, for those who have not the guiding thread which Capt. Baker has provided, to prove in some cases extremely confusing. I know learned men who succeed in producing the opposite effect and in making easy matter appear difficult.

As to the paper itself, no criticism occurs to me, but I would ask Capt. Baker whether rotation of the "cut" whereby successive planes of reflection are obtained from sections of a cylinder representing the original undeflected bundle of rays could be read off directly from the spherical diagram?

Mr C. Beck: I wish to express my appreciation of Capt. Baker's most interesting paper. It is refreshing to see a subject of this kind treated as an investigation of the general case and not confined to a particular individual application.

The existing models of the prism binocular which have two separate right angle prisms could be improved if the two prisms could be replaced by a solid prism which is convenient to make. Such a compound prism would overcome several disadvantages. It would dispense with two open surfaces which reflect light and are liable to become dirty and it would prevent damage due to the prisms being shifted in their settings. It should also be considered, especially if six reflections are investigated, whether any of the surfaces can be made to do double duty by making two reflections or one reflection and one transmission.

Mr T. Smith: Capt. Baker is to be congratulated on his lucid presentation of this subject, and his method of working will be sure to appeal to the practical man in the optical industry. Nevertheless it has to be remembered that, to those who are accustomed to use them, algebraic methods have many advantages—for instance there are no difficulties of sign—and it is simple to take into account other desirable conditions, such as those relating to minimum length of path in the glass.

Perhaps I may be permitted again to point out that the angle between two mirrors is the supplement of that marked on some of the early diagrams in this paper. It is very desirable not to perpetuate the errors associated with the use of the wrong angle, for they seem the cause of widely spread misconceptions about reflecting systems and particularly mixed reflecting and refracting systems.

Dr H. Moore: The method which Capt. Baker has just described possesses many interesting features and its potentialities would appear to be well worth exploring further. I was particularly struck by the way in which Capt. Baker made light of the various difficulties encountered and by the way in which, while he was dealing with them, he almost convinced us that they were non-existent.

Capt. Baker: Professor Filon's query is dealt with in the course of the paper.

With regard to the point raised by Mr Beck it is almost impossible to speak at present but the number of ways of designing a binocular prism system with six reflections is so great that it would be hard if among the number there are none that are worth trying.

Mr Smith's criticism that the angle between the two mirrors is the supplement of that shown on some of the early diagrams is only partly correct. The swing round the line can be made either way round the line; the angle between the two mirrors is the supplement of half the angle of swing so that the prism angle is either that shown or the supplement depending upon which swing is used. When working a practical example no confusion is likely to exist.

ON TORIC LENSES

By T. SMITH, M.A., F.Inst.P.

(Optics Department, The National Physical Laboratory.)

MS. received, 26th October, 1927. *Read and discussed, 19th January*, 1928.

ABSTRACT. A system of toric lenses having a common normal to all their surfaces possesses in general ten independent primordial coefficients. A single surface has only three degrees of freedom, and this number also holds for any system of negligible axial depth. Formulae are given for the calculation of the ten coefficients, which are only all independent when the system includes at least three separated toric refracting surfaces with their planes of principal curvature finitely inclined to one another. An eye with both its cornea and its crystalline lens astigmatic and the meridians of principal curvature different has more independent coefficients than a spectacle lens has effective degrees of freedom. The nature and importance of the unavoidable residual errors in central vision when such an eye is corrected by a toric lens are discussed.

The Degrees of Freedom of Optical Systems

When systems of lenses are classified according to the degrees of symmetry they possess, three main groups are found. In the first every possible kind of dissymmetry may exist: instruments belonging to this group have rarely, if ever, been constructed. In the second there exists a ray which meets all the lens surfaces at right angles, and the section of any one of these surfaces by a plane containing this ray is a curve symmetrical about the ray as axis. In the third group all these sections of a given surface are alike, that is to say each refracting surface is a surface of revolution having the normal ray as the axis of symmetry. The majority of optical instruments belong to the third group, and accordingly theoretical optical investigations relate preponderatingly to these axially symmetrical systems. Systems of the second group however are of special importance in ophthalmic work. The theory of these lenses is less simple than would be inferred from an examination of books dealing with this branch of optics.

The differences between typical members of the three groups as regards their optical complexity are greater than is often recognised. A simple comparison is obtained by considering the characteristics of the systems as given by algebraic expressions. Consider for instance an expression which measures the time light takes to travel from a point of a fixed plane in the object space to a point of the exit pupil in the image space. The former point requires two coordinates to specify it, say y and z; and similarly the latter point is given by two coordinates, say η and ζ. If the system belongs to the second or the third group, all these coordinates vanish for points on the normal ray. In the expression for the time, as there is no

symmetry of any kind in a system of the first group, every possible combination of the letters y, z, η, ζ raised severally to any integral power can be present. In systems of the second group, owing to the special kind of symmetry which they possess, the expression for the time taken by the light on its journey must not be altered if y, z, η, ζ are all changed in sign simultaneously. That is to say the expression for the time is an even function of these variables. We therefore obtain members of the second group from the first by putting the coefficients of all terms of odd order equal to zero. In passing to the third group we reject a number of the terms left in the second group, the rotational symmetry requiring us to retain as effective variables only the three combinations $y^2 + z^2$, $y\eta + z\zeta$, and $\eta^2 + \zeta^2$. We can now compare the typical members of the three groups by counting the number of independent finite coefficients present in the algebraic expression for the time. We will take these in sections. The first section includes terms of the first and second orders in y, z, η, ζ, and corresponds to the Gaussian imagery properties of symmetrical lenses; we will refer to these as the primordial terms: the second contains the terms of the third and fourth orders, corresponding to first order aberrations: the third consists of the fifth and sixth order terms, which in a symmetrical instrument give the second order aberrations; and so on. The numbers thus obtained are given in Table I. The outstanding fact is the remarkable simplicity of systems of the third group in comparison with those of Groups 1 and 2.

Table I

Terms	Group 1	Group 2	Group 3
*Primordial properties ...	14	10	3
First order aberrations ...	55	35	6
Second order aberrations ...	140	84	10
Third order aberrations ...	285	165	15

It is interesting to compare these figures with those giving the number of degrees of freedom of systems containing a definite number of surfaces of the appropriate types. They are obtained by taking the corresponding multiple of the number of constants in the equation of a single surface, together with one constant for the separation of each successive pair of surfaces. The number of constants per surface is given in Table II.

Table II

Terms	Group 1	Group 2	Group 3
*Primordial: first surface	5	3	1
,, each additional surface ...	6	4	2
First order aberrations per surface ...	9	5	1
Second order aberrations per surface ...	13	7	1
Third order aberrations per surface ...	17	9	1

* If the system of reference axes is chosen to reduce the number of primordial coefficients, the figures under Groups 1 and 2 in the first lines of each of Tables I and II will be altered. As the reductions in the two tables are equal the change does not affect the discussion in the text.

From the comparison of the figures in these two tables we may see the minimum number of refracting surfaces an instrument must have before it can exhibit all the degrees of freedom possible in a system of its group. Taking first the primordial properties, we see that a single thick lens, that is a system of two separated surfaces, is a typical member of Group 3, but in Groups 1 and 2 at least three surfaces are necessary. For first order aberrations of given primordial construction six surfaces suffice for Group 3, but seven are needed for Groups 1 and 2. For second order aberrations the numbers appear to be 11, 12, and 10, and for third order aberrations 17, 19, and 15 for Groups 1, 2, and 3 respectively. The figures for Group 1 need slight revision; the correct numbers are equal to those of Group 2; the false result is due to the combination of two heterogeneous classes of terms, those of even and those of odd orders, to yield the figures tabled for this group. At present we are concerned with Group 2, and the point to which attention is directed is that it is impossible for a lens having only two refracting surfaces to possess the full degrees of freedom of the lowest order which are typical of such systems. On the other hand systems in which three refracting surfaces are present may have all these degrees of freedom. To take examples, we have not independent control of all the properties with a spectacle lens; on the other hand in an eye, since refraction occurs at the cornea and at each surface of the crystalline lens, any conceivable combination of properties is theoretically possible. The number of faults calling for correction in an eye may thus exceed those which it is possible for us to correct with any lens having two surfaces. Whether human eyes exist in which such an inconsistency appears in an acute form cannot be considered here. It is enough to have indicated that a theory, in which the subject is deemed to have been exhausted when the properties of a system have been referred to the refracting powers in two principal meridians at right angles to one another, together with a rotation about the axis in which these planes meet, is not unlikely to prove inadequate.

The Laws of Combination of the Primordial Coefficients

We have seen that the full complexity of the typical system is only acquired gradually as surface is added to surface. This suggests that a convenient method of investigating the properties of such systems is to express the coefficients of a combination of surfaces in terms of those of two sub-groups into which the combination may be divided, either sub-group consisting of one or more surfaces. The eikonal, which can readily be constructed for any refracting surface, will be used as the basis of the investigation.

Suppose then that the terms of the lowest order in the eikonal of the first portion of the system are

$$\tfrac{1}{2}aM^2 + \tfrac{1}{2}bN^2 + \tfrac{1}{2}cM'^2 + \tfrac{1}{2}dN'^2 + eMN + fM'N' - gMN' - hM'N - iMM' - jNN',$$

those for the second part

$$\tfrac{1}{2}a'M'^2 + \tfrac{1}{2}b'N'^2 + \tfrac{1}{2}c'M''^2 + \tfrac{1}{2}d'N''^2 + e'M'N' + f'M''N'' - g'M'N'' - h'M''N' - i'M'M'' - j'N'N'',$$

and similarly those for the complete system

$$\tfrac{1}{2}AM^2 + \tfrac{1}{2}BN^2 + \tfrac{1}{2}CM''^2 + \tfrac{1}{2}DN''^2 + EMN + FM''N'' - GMN'' - HM''N - IMM'' - JNN'' \qquad (1),$$

where the directions of a ray in the object space, between the two components, and in the image space are given respectively by the pairs of direction cosines M, N; M', N'; and M'', N''. The eikonal for the complete system is obtained by adding together those of its parts and giving the intermediate variables M', N' the values which make the sum stationary, i.e. the values which satisfy

$$(c + a')\,M' + (f + e')\,N' = iM + hN + i'M'' + g'N'',$$
$$(f + e')\,M' + (d + b')\,N' = gM + jN + h'M'' + j'N''.$$

In this way we find

$$\left.\begin{aligned}
A - a &= -i^2u + 2igv - g^2w \\
B - b &= -h^2u + 2hjv - j^2w \\
C - c' &= -i'^2u + 2i'h'v - h'^2w \\
D - d' &= -g'^2u + 2g'j'v - j'^2w \\
E - e &= -ihu + (hg + ij)\,v - gjw \\
F - f' &= -i'g'u + (g'h' + i'j')\,v - h'j'w \\
G &= ig'u - (g'g + ij')\,v + gj'w \\
H &= hi'u - (hh' + i'j)\,v + jh'w \\
I &= ii'u - (i'g + ih')\,v + gh'w \\
J &= g'hu - (g'j + hj')\,v + jj'w
\end{aligned}\right\} \qquad \text{.....(2),}$$

where

$$\frac{u}{d + b'} = \frac{v}{f + e'} = \frac{w}{c + a'}$$

$$= \frac{1}{(c + a')\,(d + b') - (f + e')^2} = uw - v^2 \qquad \text{.....(3).}$$

Special Systems

These laws of combination hold whether the separation into two parts takes place in glass or in air. They are also unaltered in form (although the value of the coefficients is then changed) if instead of denoting by M, N, etc., the actual direction cosines of a ray, they are any multiples of them. It is convenient to let them represent the product of the direction cosines with the appropriate refractive index. Under these conditions it is known that the eikonal of the surface

$$x = \tfrac{1}{2}\alpha y^2 - \beta yz + \tfrac{1}{2}\gamma z^2$$

is

$$\frac{\gamma\,(M' - M)^2 + 2\beta\,(M' - M)\,(N' - N) + \alpha\,(N' - N)^2}{2\,(\alpha\gamma - \beta^2)\,(\mu' - \mu)} + \ldots.$$

In other words the characteristics of single surface refraction are

$$\left.\begin{aligned} a &= c = i \\ b &= d = j \\ e &= f = g = h \end{aligned}\right\} \qquad \ldots\ldots(4),$$

and if the lines of principal curvature at the origin coincide with the axes of y and z, the last four quantities are all zero.

If we combine any two systems which have

$$e = f = g = h = e' = f' = g' = h' = 0,$$

it is evident that $v = 0$, and the equations then show that for the combination

$$E = F = G = H = 0.$$

These conditions then may be secured for any system in which the lines of principal curvature at the axis of all the refracting surfaces lie in the same pair of planes. In particular, if the system contains only one surface of double curvature, the axes may be so chosen as to make these four quantities simultaneously zero. Thus if the crystalline lens of the eye is effectively symmetrical about an axis, we should expect to find that satisfactory correction for astigmatism could be effected by the use of a spectacle lens having one surface of double curvature.

The equalities

$$\begin{aligned} a &= c = i, \\ b &= d = j \end{aligned}$$

will be disturbed when the axial reference point is moved. If the point is moved through a distance t the change in path is $\mu t \cos\psi$ where ψ is the angle made by the ray with the axis. If the change is in the object space this distance is deducted provided the displacement is in the positive direction, and if in the image space it is added. Since

$$\mu t \cos\psi \doteq \mu t - \frac{t}{2\mu}(\mu \sin\psi)^2 = \mu t - \frac{\tau}{2}(\mu \sin\psi)^2,$$

such a change in the object space is equivalent to the addition of t/μ or τ to a and b, and if in the image space to the subtraction of τ from c and d. Changes of origin, it will be observed, cannot affect e, f, g, h, i, and j.

From the general formulae we write down expressions for the differences between the general constants; it is only necessary for us to quote two, as the others all follow from symmetry. We have

$$A - I = u\{a(c+a') - i(i+i')\} + v\{i(g+h') - a(f+e')\} + g\{v(i+i') - w(g+h')\},$$
$$E - H = u\{e(c+a') - h(i+i')\} + v\{h(g+h') - e(f+e')\} + j\{v(i+i') - w(g+h')\}.$$

Apply these formulae to a system of two refracting surfaces separated by an axial distance t. Then

$$\begin{aligned} e &= f = g = h, \\ e' &= f' = g' = h', \\ a &= i,\ b = j,\ c' = i',\ d' = j', \end{aligned}$$

and

$$c + a' = i + i' - \tau,\ d + b' = j + j' - \tau.$$

We therefore find

$$\left.\begin{aligned} A - I &= -(iu - gv)\tau \\ E - H &= -(gu - jv)\tau \end{aligned}\right\} \quad \ldots\ldots(5),$$

and from these and the corresponding equations it follows that as the separation between the surfaces is diminished we approach continuously to the conditions

$$A = C = I,$$
$$B = D = J,$$
$$E = F = G = H.$$

In other words a thin lens (and hence also a combination of any number of thin lenses in axial contact) has all the properties characteristic of a single surface.

Correction of Visual Defects

The conclusion we have now reached is that the current theory of toric lenses is correct when the planes of principal curvature are fixed and when all axial thicknesses are neglected. In other cases it is incorrect. It is almost certain that the properties proved for a lens of zero thickness have been supposed to be applicable qualitatively to any system. The methods in use for the transposition of oculists' prescriptions are based upon this thin lens theory. Since all ordinary spectacle lenses are approximately thin the errors made in this way will undoubtedly be negligible.

We are not yet entitled to draw much comfort from this conclusion, however pleasing it may be to the maker of spectacle lenses. The improved position of the current theory in its application to spectacle lenses involves as an inevitable corollary decreased control over the properties of the compound eye-plus-spectacle system. We may in fact find it still less possible than we had anticipated from the enumeration of the constants of the system to secure full correction of the defects of what we may call the "skew" eye. Since the separations between the three refracting surfaces of the eye as well as the distance between the cornea and the correcting lens are by no means negligible in comparison with the focal length of the eye, the conclusion appears inevitable that our lenses are incapable of securing mathematically exact correction of astigmatism if differently oriented meridians can occur in both the cornea and the crystalline lens. We have always however to remember that on physical grounds a not inappreciable departure from the strict geometrical conditions of good imagery is permissible. To discuss this question we require to know both the nature and the extent of the departures from the ideal conditions which may be present.

Let us suppose that A, B, C, ... in equation (1) are the constants of the eye-plus-spectacle system. We are of course assuming that the eye is fixed both in position and in accommodation; to be definite we will suppose that distant vision is in question. Then for light incident parallel to the normal ray $M = N = 0$, and the variation in path is represented by

$$\tfrac{1}{2}CM''^2 + \tfrac{1}{2}DN''^2 + FM''N''.$$

If the reference point in the image space is moved forward by the reduced distance X, the variation becomes

$$\tfrac{1}{2}(C - X)M''^2 + \tfrac{1}{2}(D - X)N''^2 + FM''N''.$$

We wish this to be zero for all values of M'' and N''. The conditions to be secured are therefore

$$C = D = X, \quad F = 0 \qquad \ldots\ldots(6).$$

All these are quite definite conditions, for the distance X is a property of the eye. If the calculations for the eye are carried out in a normal way X will be the reduced distance from the posterior surface of the crystalline lens to the retina. Referring now to equations (2) and regarding a, b, c, ... as the constants of the spectacle lens and a', b', c', ... as those of the eye, we see that the constants of the eyeglass only enter into the expressions for C, D, and F through the quantities u, v, w. Thus from equations (3) the desired values of C, D, and F are secured by finding correct values for c, d, and f. Since these are the three quantities over which we have independent control in the spectacle lens we can always satisfy these three conditions. That is to say, since the whole of this argument is based on the assumption that $M = N = 0$, we have shown that all the degrees of freedom of the thin toric lens will be exhausted in securing a good image of a single point.

The user of spectacles however is not likely to be satisfied with a lens which only gives him distinct vision of a single point. In any given configuration of the complete optical system at least a small central area must be clearly seen. We have then to consider the nature of the image seen when M and N have small finite values. Let y' and z' be the coordinates of the point in which the ray M, N, M', N' meets the retina. Then by a well-known property of the eikonal

$$y' = IM + HN - (C - X)M'' - FN'',$$
$$z' = GM + JN - FM'' - (D - X)N''.$$

Thus when light is brought correctly to a focus on the retina for a single point, all other distant points will produce well-focussed images in the positions given by

$$y' = IM + HN, \quad z' = GM + JN \qquad \ldots\ldots(7).$$

Now if the image were a simple projection of the object, as would be the case were no astigmatism present, we should have

$$I = J, \quad G = H = 0 \qquad \ldots\ldots(8).$$

Any inequality between I and J indicates that the scale on which external objects are depicted on the retina changes with the direction of measurement. Superposed on this is another defect which can only disappear when G and H are equal to one another. This fault is most simply regarded as angular distortion.

For perfect vision we thus require to add three other conditions to the three which we found our spectacle lens would enable us to secure. The first set of three conditions has to be satisfied to yield satisfactory monocular vision. The three conditions of the second set, on the other hand, are likely to be of most consequence

on account of the binocular nature of normal vision. If a "skew" eye and one of simpler properties have to work in combination, there are always present two distinct distortions of the picture seen by one eye in comparison with that observed by the other eye. Since the sharply focussed field of view of which we are conscious at any given moment is small it may prove that such distortions are not serious enough to be of practical account. At the moment no numerical data sufficient to provide an answer to the question are known to the author, and the magnitude of the probable effects can be best appreciated from expressions which contain only the fundamental constants of the three refracting surfaces to which we assume the eye is equivalent, together with their reduced separations.

Alternative Formulae for the Primordial Coefficients of a Compound System

Before carrying out the calculations involved in the construction of these expressions we will obtain alternative formulae for the computation of the primordial coefficients. It will be observed that in (2) the magnitudes denoted by the letters are lengths with the exception of the three letters u, v, w, which signify curvatures. These are not however coefficients of any element of the lens, but auxiliary quantities introduced as a mere convenience in setting down the formulae. There are good reasons for preferring the general optical practice of taking powers rather than lengths as the fundamental quantities connected with refraction, the only lengths entering the formulae being the reduced separations between the individual refracting surfaces. We therefore seek to construct formulae of this type.

If we arrange the coefficients in order according to the variables associated with them we obtain the arrangement

A	E	I	G
E	B	H	J
I	H	C	F
G	J	F	D

We suppose this to consist of two separate parts, the division being made by a horizontal section between the second and third rows. If now P is the determinant obtained by taking any two columns in the upper portion of this arrangement, the formulae (2) give, if the leading part of the combination is thin,

$$P = (uw - v^2)(ij - gh)p',$$

where p' is obtained from P by writing small accented letters in place of capitals, with the exception that $a' - \tau$ and $b' - \tau$ are written in place of A and B respectively. Similarly, if Q is the determinant formed from any two columns of the lower portion of the square array, and if the final part of the combination is thin, we have

$$Q = (uw - v^2)(i'j' - g'h')q,$$

where q is obtained by writing small unaccented letters in Q instead of capitals, and $c-\tau$, $d-\tau$ in place of C and D respectively. It will be observed that both P and Q may represent six quantities, but as we have one identical relation between the six of each set, we may without loss reject one of each set for purposes of calculation. We will choose to neglect the determinants $|\,AB\,|$ and $|\,CD\,|$. Since $|\,IJ\,|$ is a member of both sets we then have nine distinct determinants. Now the relation found for $|\,IJ\,|$ is

$$|\,IJ\,| = (uw - v^2)\,|\,ij\,||\,i'j'\,|\,,$$

and this may be used to divide all the other relations we have found. We then have four relations represented by

$$\frac{P}{|\,IJ\,|} = \frac{p'}{|\,i'j'\,|}$$

and four by

$$\frac{Q}{|\,IJ\,|} = \frac{q}{|\,ij\,|}.$$

It may be observed that for a thin system P and Q will vanish if they are constructed by taking the two columns occupying the two odd or the two even positions in the original array; and in every other case $P = Q = |\,IJ\,|$. Further the formulae we have found are evidently adapted to the evaluation of A, B, E and C, D, F respectively; thus it will not be essential to evaluate more than three members of each group.

The quotients on the right of the last two equations differ from those on the left in the presence in the numerator of τ, the reduced separation between the end surface and its neighbour. The coefficients of τ evidently always belong to the group

$$\frac{G}{|\,IJ\,|},\ \frac{H}{|\,IJ\,|},\ \frac{I}{|\,IJ\,|},\ \frac{J}{|\,IJ\,|}.$$

We can therefore complete the scheme of calculations by finding formulae for these quantities. Two expressions for each, corresponding to the two directions in which the calculations may be made, are easily obtained from equations (2).

The suitability of $|\,IJ\,|$ as a divisor can be seen by noting that it represents the product of the principal radii of curvature of the wave-front in single surface refraction: H, I, and J are linear functions of these radii, so that division by $|\,IJ\,|$ yields linear functions of the curvatures. It should be observed that both $|\,IJ\,|$ and $(I+J)$ are unaltered as the surface is rotated, so that $I+J$ represents the sum of the principal radii and $\dfrac{I+J}{|\,IJ\,|}$ the sum of the principal curvatures. On the other hand H is proportional to the difference of the radii: in fact $\frac{1}{2}\tan^{-1}\left(\dfrac{2H}{I \sim J}\right)$ is the angle through which the surface has been rotated from the position in which the lines of principal curvature lie in the planes $y = 0$ and $z = 0$.

The working formulae will now be arranged in two groups, one suitable for working through the system from the first surface towards the last, the other for

progress in the reverse direction. In terms of the notation used hitherto the meanings of the new symbols are as follow:

$$\left.\begin{aligned} \mathscr{A} &= \frac{|AJ|}{|IJ|}, \quad \mathscr{B} = \frac{|IB|}{|IJ|}, \quad \mathscr{C} = \frac{|CJ|}{|IJ|}, \quad \mathscr{D} = \frac{|ID|}{|IJ|} \\ \mathscr{G} &= \frac{|GB|}{|IJ|}, \quad \mathscr{I} = \frac{|AH|}{|IJ|}, \quad \mathscr{U} = \frac{|HD|}{|IJ|}, \quad \mathscr{V} = \frac{|CG|}{|IJ|} \\ \mathscr{E} &= \frac{G}{|IJ|}, \quad \mathscr{H} = \frac{H}{|IJ|}, \quad \mathscr{I} = \frac{I}{|IJ|}, \quad \mathscr{J} = \frac{J}{|IJ|} \end{aligned}\right\} \quad \text{.....(9).}$$

These give a new array of variables

$$\begin{array}{cccc} \mathscr{A} & \mathscr{I} & \mathscr{I} & \mathscr{G} \\ \mathscr{G} & \mathscr{B} & \mathscr{H} & \mathscr{J} \\ \mathscr{I} & \mathscr{H} & \mathscr{C} & \mathscr{V} \\ \mathscr{E} & \mathscr{J} & \mathscr{U} & \mathscr{D} \end{array}$$

in which distinct variables $\mathscr{G}$ and $\mathscr{I}$ take the place of the two E's, and similarly $\mathscr{U}$ and $\mathscr{V}$ replace the two F's. The old variables may be obtained from the relations

$$\left.\begin{aligned} &\frac{A}{\mathscr{A}\mathscr{J}-\mathscr{I}\mathscr{I}} = \frac{B}{\mathscr{B}\mathscr{J}-\mathscr{G}\mathscr{H}} = \frac{C}{\mathscr{C}\mathscr{I}-\mathscr{V}\mathscr{H}} = \frac{D}{\mathscr{D}\mathscr{J}-\mathscr{U}\mathscr{I}} \\ &= \frac{E}{\mathscr{A}\mathscr{H}-\mathscr{I}\mathscr{J}} = \frac{E}{\mathscr{B}\mathscr{I}-\mathscr{G}\mathscr{I}} = \frac{F}{\mathscr{C}\mathscr{G}-\mathscr{V}\mathscr{J}} = \frac{F}{\mathscr{D}\mathscr{H}-\mathscr{U}\mathscr{I}} \\ &= \frac{G}{\mathscr{E}} = \frac{H}{\mathscr{H}} = \frac{I}{\mathscr{I}} = \frac{J}{\mathscr{J}} = \frac{1}{\mathscr{I}\mathscr{J}-\mathscr{I}\mathscr{H}} \end{aligned}\right\} \quad \text{(10).}$$

The equations which will normally be used in the forward direction are

$$\left.\begin{aligned} \mathscr{C}_{1,n+1} &= \mathscr{C}_{1,n} - \mathscr{I}_{1,n}\tau_n \\ \mathscr{U}_{1,n+1} &= \mathscr{U}_{1,n} - \mathscr{H}_{1,n}\tau_n \\ \mathscr{H}_{1,n+1} &= \mathscr{H}_{1,n} + \mathscr{G}_{n+1}\mathscr{C}_{1,n+1} + \mathscr{I}_{n+1}\mathscr{U}_{1,n+1} \\ \mathscr{I}_{1,n+1} &= \mathscr{I}_{1,n} + \mathscr{J}_{n+1}\mathscr{C}_{1,n+1} + \mathscr{H}_{n+1}\mathscr{U}_{1,n+1} \\ \mathscr{D}_{1,n+1} &= \mathscr{D}_{1,n} - \mathscr{I}_{1,n}\tau_n \\ \mathscr{V}_{1,n+1} &= \mathscr{V}_{1,n} - \mathscr{G}_{1,n}\tau_n \\ \mathscr{G}_{1,n+1} &= \mathscr{G}_{1,n} + \mathscr{H}_{n+1}\mathscr{D}_{1,n+1} + \mathscr{J}_{n+1}\mathscr{V}_{1,n+1} \\ \mathscr{I}_{1,n+1} &= \mathscr{I}_{1,n} + \mathscr{I}_{n+1}\mathscr{D}_{1,n+1} + \mathscr{I}_{n+1}\mathscr{V}_{1,n+1} \end{aligned}\right\} \quad \text{.....(11),}$$

and it will be noted that these fall into two independent groups of four equations each. The $\mathscr{G}$ and $\mathscr{H}$ of each single surface are equal to one another, and the above forms, with separate letters for these quantities, are adopted to preserve the symmetry of the expressions. The initial data of the system consist of course of the values of $\mathscr{G}$, $\mathscr{H}$, $\mathscr{I}$, $\mathscr{J}$ for each surface and of τ_n, the reduced separation between surfaces n and $n+1$. In addition we have the initial values

$$\left.\begin{aligned} \mathscr{A}_{p,p} &= \mathscr{B}_{p,p} = \mathscr{C}_{p,p} = \mathscr{D}_{p,p} = 1 \\ \mathscr{G}_{p,p} &= \mathscr{I}_{p,p} = \mathscr{U}_{p,p} = \mathscr{V}_{p,p} = 0 \end{aligned}\right\} \quad \text{.....(12).}$$

Formulae of the forward type may also be obtained for $\mathscr{A}, \mathscr{B}, \mathscr{G}, \mathscr{I}$ for a series of surfaces, but it will normally be better to use the reverse formulae

$$\left.\begin{aligned}
\mathscr{A}_{n,l} &= \mathscr{A}_{n+1,l} - \mathscr{I}_{n+1,l}\tau_n \\
\mathscr{G}_{n,l} &= \mathscr{G}_{n+1,l} - \mathscr{G}_{n+1,l}\tau_n \\
\mathscr{G}_{n,l} &= \mathscr{G}_{n+1,l} + \mathscr{H}_n \mathscr{A}_{n,l} + \mathscr{I}_n \mathscr{G}_{n,l} \\
\mathscr{I}_{n,l} &= \mathscr{I}_{n+1,l} + \mathscr{J}_n \mathscr{A}_{n,l} + \mathscr{G}_n \mathscr{G}_{n,l} \\
\mathscr{B}_{n,l} &= \mathscr{B}_{n+1,l} - \mathscr{I}_{n+1,l}\tau_n \\
\mathscr{I}_{n,l} &= \mathscr{I}_{n+1,l} - \mathscr{H}_{n+1,l}\tau_n \\
\mathscr{H}_{n,l} &= \mathscr{H}_{n+1,l} + \mathscr{G}_n \mathscr{B}_{n,l} + \mathscr{J}_n \mathscr{I}_{n,l} \\
\mathscr{I}_{n,l} &= \mathscr{I}_{n+1,l} + \mathscr{I}_n \mathscr{B}_{n,l} + \mathscr{H}_n \mathscr{I}_{n,l}
\end{aligned}\right\} \quad \ldots\ldots(13),$$

where l denotes the last surface of the system. If the complete set of coefficients is computed we may note that we have six independent checks, viz. two evaluations of $\mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}$ and the equalities

$$\mathscr{A}\mathscr{H} - \mathscr{I}\mathscr{J} = \mathscr{B}\mathscr{G} - \mathscr{G}\mathscr{I}, \quad \mathscr{C}\mathscr{G} - \mathscr{V}\mathscr{J} = \mathscr{D}\mathscr{H} - \mathscr{U}\mathscr{I}.$$

Since these involve all the coefficients of the system these relations should suffice to ensure the accuracy of numerical work. The equations we have just found correspond exactly to those found most convenient in evaluating the Gaussian coefficients of a system belonging to Group 3. From this analogy it is clear that the coefficients $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{G}, \mathscr{I}, \mathscr{U}, \mathscr{V}$ relate to the positions of the external foci, and that $\mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}$ are four "powers." That a general toric system has four powers, not merely the two usually considered, is the real meaning of equations (7).

Evaluation of Outstanding Visual Coefficients

We are now in a position to resume the consideration of the outstanding defects which may be present in the eye, represented by the values of $I - J$, G, H. We notice in the first place that instead of G, H, I, J we may consider $\mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}$ which are proportional to them. Secondly we observe that the conditions

$$C = D = X, \quad F = 0$$

are equivalent to

$$\frac{\mathscr{C}}{\mathscr{J}} = \frac{\mathscr{D}}{\mathscr{I}} = \frac{\mathscr{U}}{\mathscr{H}} = \frac{\mathscr{V}}{\mathscr{G}} = X.$$

Thirdly, calculation by the formulae for $\mathscr{C}, \mathscr{D}, \ldots$ involves the three variables of the correcting spectacle lens, and these it is desired to exclude. The difficulty is met by using the equations in the reverse direction to express $\mathscr{G}_{1,0}, \mathscr{H}_{1,0}, \mathscr{I}_{1,0}, \mathscr{J}_{1,0}$ in terms of $\mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}$, and applying the conditions

$$\left(S - \frac{\partial}{\partial\phi_1}\right)\mathscr{G}_{1,0} = \left(S - \frac{\partial}{\partial\phi_1}\right)\mathscr{H}_{1,0} = 0,$$

$$\left(S - \frac{\partial}{\partial\phi_1}\right)\mathscr{I}_{1,0} = \left(S - \frac{\partial}{\partial\psi_1}\right)\mathscr{J}_{1,0} = 1,$$

$$\mathscr{G}_{1,0} = \mathscr{H}_{1,0},$$

which represent the fact that the spectacle lens is equivalent to a single toric surface (here indicated by the suffix o), where S is the distance between the spectacle and the cornea. The four apparent single surface variables may be replaced by the three actual variables by writing

$$\mathscr{S}_n = \mathscr{H}_n = \pi_n, \quad \mathscr{I}_n = \phi_n, \quad \mathscr{J}_n = \psi_n.$$

The process indicated is very easily carried out. We find for the extended expressions from the first group of four equations

$$\begin{aligned}
\mathscr{H}_{1,0} = {} & \mathscr{H}[1 - \phi_1(\tau_1+\tau_2+X) - \phi_2(\tau_2+X) - \phi_3 X + \phi_1\tau_1\phi_2(\tau_2+X) \\
& + \phi_1(\tau_1+\tau_2)\phi_3 X + \phi_2\tau_2\phi_3 X - \phi_1\tau_1\phi_2\tau_2\phi_3 X \\
& + \pi_1\tau_1\pi_2(\tau_2+X) + \pi_1(\tau_1+\tau_2)\pi_3 X + \pi_2\tau_2\pi_3 X \\
& - \phi_1\tau_1\pi_2\tau_2\pi_3 X - \pi_1\tau_1\psi_2\tau_2\pi_3 X - \pi_1\tau_1\pi_2\tau_2\phi_3 X] \\
& - \mathscr{I}[\pi_1(\tau_1+\tau_2+X) + \pi_2(\tau_2+X) + \pi_3 X + \pi_1\tau_1\pi_2\tau_2\pi_3 X \\
& - \pi_1\tau_1\psi_2(\tau_2+X) - \phi_1\tau_1\pi_2(\tau_2+X) - \pi_1(\tau_1+\tau_2)\psi_3 X - \phi_1(\tau_1+\tau_2)\pi_3 X \\
& - \pi_2\tau_2\psi_3 X - \phi_2\tau_2\pi_3 X + \pi_1\tau_1\psi_2\tau_2\psi_3 X \\
& + \phi_1\tau_1\pi_2\tau_2\psi_3 X + \phi_1\tau_1\phi_2\tau_2\pi_3 X], \\
\mathscr{I}_{1,0} = {} & \mathscr{I}[1 - \psi_1(\tau_1+\tau_2+X) - \psi_2(\tau_2+X) - \psi_3 X + \psi_1\tau_1\psi_2(\tau_2+X) \\
& + \psi_1(\tau_1+\tau_2)\psi_3 X + \psi_2\tau_2\psi_3 X - \psi_1\tau_1\psi_2\tau_2\psi_3 X \\
& + \pi_1\tau_1\pi_2(\tau_2+X) + \pi_1(\tau_1+\tau_2)\pi_3 X + \pi_2\tau_2\pi_3 X \\
& - \psi_1\tau_1\pi_2\tau_2\pi_3 X - \pi_1\tau_1\phi_2\tau_2\pi_3 X - \pi_1\tau_1\pi_2\tau_2\psi_3 X] \\
& - \mathscr{H}[\pi_1(\tau_1+\tau_2+X) + \pi_2(\tau_2+X) + \pi_3 X + \pi_1\tau_1\pi_2\tau_2\pi_3 X \\
& - \pi_1\tau_1\phi_2(\tau_2+X) - \psi_1\tau_1\pi_2(\tau_2+X) - \pi_1(\tau_1+\tau_2)\phi_3 X - \psi_1(\tau_1+\tau_2)\pi_3 X \\
& - \pi_2\tau_2\phi_3 X - \psi_2\tau_2\pi_3 X + \pi_1\tau_1\phi_2\tau_2\phi_3 X \\
& + \psi_1\tau_1\pi_2\tau_2\phi_3 X + \psi_1\tau_1\psi_2\tau_2\pi_3 X],
\end{aligned}$$

with corresponding expressions for $\mathscr{S}_{1,0}$ and $\mathscr{I}_{1,0}$ from the four equations of the second group. It will be noticed that the coefficients of $\mathscr{J}$ and $\mathscr{H}$ in $\mathscr{J}_{1,0}$ are obtainable from those of $\mathscr{H}$ and $\mathscr{J}$ in $\mathscr{H}_{1,0}$ by interchanging ϕ and ψ throughout. It follows that the coefficients of $\mathscr{I}$ and $\mathscr{S}$ in $\mathscr{I}_{1,0}$ are equal to those of $\mathscr{H}$ and $\mathscr{J}$ in $\mathscr{H}_{1,0}$, and those of $\mathscr{S}$ and $\mathscr{I}$ in $\mathscr{S}_{1,0}$ are equal to those of $\mathscr{J}$ and $\mathscr{H}$ in $\mathscr{J}_{1,0}$ respectively. This conclusion might have been derived at once from the general equations. It follows that the evaluation of a single function, say $\mathscr{H}_{1,0}$, would have enabled us to find all we want to know. It could also have been inferred that the first coefficient on the right of each equation must be an even function of the π's, and the second an odd function of these variables. The identities among the coefficients that we have just noted enable us to write

$$\begin{aligned}
\mathscr{S}_{1,0} &= \mathscr{S}r - \mathscr{I}p, \\
\mathscr{H}_{1,0} &= \mathscr{H}s - \mathscr{J}q, \\
\mathscr{I}_{1,0} &= \mathscr{I}s - \mathscr{S}q, \\
\mathscr{J}_{1,0} &= \mathscr{J}r - \mathscr{H}p,
\end{aligned}$$

and the conditions satisfied by the quantities on the left lead at once to

$$G = \left(S - \frac{\partial}{\partial\psi_1}\right) p,$$

$$H = \left(S - \frac{\partial}{\partial\phi_1}\right) q,$$

$$I = \left(S - \frac{\partial}{\partial\psi_1}\right) r,$$

$$J = \left(S - \frac{\partial}{\partial\phi_1}\right) s.$$

Thus

$$\begin{aligned} G = {} & S\pi_1(\tau_1+\tau_2+X) + (S+\tau_1)\pi_2(\tau_2+X) + (S+\tau_1+\tau_2)\pi_3 X + S\pi_1\tau_1\pi_2\tau_2\pi_3 X \\ & - S\pi_1\tau_1\phi_2(\tau_2+X) - S\psi_1\tau_1\pi_2(\tau_2+X) - S\pi_1(\tau_1+\tau_2)\phi_3 X - S\psi_1(\tau_1+\tau_2)\pi_3 X \\ & - (S+\tau_1)\pi_2\tau_2\phi_3 X - (S+\tau_1)\psi_2\tau_2\pi_3 X + S\pi_1\tau_1\phi_2\tau_2\phi_3 X \\ & + S\psi_1\tau_1\pi_2\tau_2\phi_3 X + S\psi_1\tau_1\psi_2\tau_2\pi_3 X, \end{aligned}$$

$$\begin{aligned} I = {} & S+\tau_1+\tau_2+X - S\psi_1(\tau_1+\tau_2+X) - (S+\tau_1)\psi_2(\tau_2+X) - (S+\tau_1+\tau_2)\psi_3 X \\ & + S\psi_1\tau_1\psi_2(\tau_2+X) + S\psi_1(\tau_1+\tau_2)\psi_3 X + (S+\tau_1)\psi_2\tau_2\psi_3 X - S\psi_1\tau_1\psi_2\tau_2\psi_3 X \\ & + S\pi_1\tau_1\pi_2(\tau_2+X) + S\pi_1(\tau_1+\tau_2)\pi_3 X + (S+\tau_1)\pi_2\tau_2\pi_3 X \\ & - S\psi_1\tau_1\pi_2\tau_2\pi_3 X - S\pi_1\tau_1\phi_2\tau_2\pi_3 X - S\pi_1\tau_1\pi_2\tau_2\psi_3 X, \end{aligned}$$

with expressions for H and J obtained by writing ϕ, ψ in place of ψ, ϕ.

Now the astigmatism of the individual surfaces is measured in two perpendicular planes by π and $\phi - \psi$. It follows that $I - J$ contains astigmatic terms only of degrees one and three. Since the astigmatism may be assumed small compared with the normal refraction, only the terms of degree one need be retained. Again $G - H$ is of the second degree in the astigmatism, and we shall therefore not be seriously in error in assuming $G = H$. We are then enabled to write

$$G \doteq H \doteq \Sigma\Delta_n\pi_n\Delta'_n,$$

$$\tfrac{1}{2}(I - J) \doteq \Sigma\Delta_n\varpi_n\Delta'_n,$$

where ϖ is written for $\phi - \psi$, so that π and ϖ are perpendicular components of the astigmatism, and Δ_n, Δ'_n denote the reduced optical distances of the nth surface of the eye from the spectacle lens and from the retina respectively. These optical distances are to be calculated on the assumption that the refracting surfaces of the eye are spherical; we may therefore calculate the products $\Delta_n\Delta'_n$ once for all, using such values as those given by Helmholtz and Gullstrand for the schematic eye. These quantities will be most usefully compared with $\frac{1}{2}(I + J)$, also computed on the assumption that the surfaces are spherical. The calculations have been made for a schematic eye of five refracting surfaces, the additional surfaces belonging to the "equivalent core lens" of Gullstrand*. It is obvious that the approximate expressions we have found will hold for a system of more than three surfaces. The values found for the coefficients are given in Table III.

* Helmholtz, *Physiological Optics*. Translated by J. P. C. Southall, vol. I, p. 351.

The approximate equality of these coefficients enables us to say that we obtain the values of G and H or of $\frac{1}{2}(I-J)$ in millimetres by dividing the sum of the resolved parts of the astigmatism of all the individual surfaces of the eye expressed in dioptres, by six. Thus with 1 D of astigmatism the magnification in one direction will be greater, and in the perpendicular direction will be less, by 1 per cent. than the mean value.

Table III

$\frac{1}{2}(I+J)$	=	16·96 mm.
$\Delta\Delta'$ (cornea)	=	170 sq. mm.
$\Delta\Delta'$ (anterior lens surface)	=	171 ,,
$\Delta\Delta'$ (anterior core surface)	=	170 ,,
$\Delta\Delta'$ (posterior core surface)	=	163 ,,
$\Delta\Delta'$ (posterior lens surface)	=	161 ,,

It now only remains to show that $G-H=0$ implies the absence of angular distortion, i.e. that we can find two perpendicular directions in the object space which are imaged by lines in the same two directions, or that by a rotation of the system we can secure the disappearance of the terms in MN'' and $M''N$ from the eikonal. To show this we substitute $M\cos\theta - N\sin\theta$, $M\sin\theta + N\cos\theta$, $M''\cos\theta - N''\sin\theta$, $M''\sin\theta + N''\cos\theta$ for M, N, M'', N'' respectively in the eikonal term given by (1). The transformed expression, when the additional terms in X are added, is

$$\begin{aligned}
&\tfrac{1}{2}\left[\tfrac{1}{2}(A+B)(M^2+N^2)-(G-H)(MN''-M''N)-(I+J)(MM''+NN'')\right.\\
&\qquad\left.+\tfrac{1}{2}(C+D-2X)(M''^2+N''^2)\right]\\
&+\tfrac{1}{2}\cos 2\theta\left[\tfrac{1}{2}(A-B)(M^2-N^2)+2EMN-(G+H)(MN''+M''N)-(I-J)(MM''-NN'')\right.\\
&\qquad\left.+2FM''N''+\tfrac{1}{2}(C-D)(M''^2-N''^2)\right]\\
&+\tfrac{1}{2}\sin 2\theta\left[E(M^2-N^2)-(A-B)MN+(I-J)(MN''+M''N)-(G+H)(MM''-NN'')\right.\\
&\qquad\left.-(C-D)M''N''+F(M''^2-N''^2)\right].
\end{aligned}$$

The terms appearing on the second line within each square bracket disappear in virtue of the relations $C=D=X$, $F=0$. If $G=H$ the term in $(MN''-M''N)$ in the first bracket vanishes, and the terms in $(MN''+M''N)$ in the second and third brackets will disappear if θ is chosen to satisfy

$$\tan 2\theta = \frac{G+H}{I-J}.$$

The remaining terms are now of the form

$$\tfrac{1}{2}A'M^2+\tfrac{1}{2}B'N^2+E'MN-I'MM''-J'NN'',$$

that is to say $G'=H'=0$, and the ray (M, N, M'', N'') meets the retina in the point

$$y'=I'M,\ z'=J'N.$$

In other words the images of distant lines parallel to the axes of coordinates are parallel to the corresponding object lines.

The conclusions to which we are led are thus that when astigmatism has been corrected by ensuring a sharply focussed image on the retina, the magnification of extended objects will in general vary with the direction of measurement. It will

be very nearly the case that the position of the image of any point can be expressed in terms of the maximum and minimum magnifications, which relate to two directions at right angles to one another. The difference between these two extreme magnifications will not ordinarily be great enough to impair binocular vision.

It has been shown that the angular distortion, which is theoretically present in "skew" eyes, is of the second order in the astigmatism of the component surfaces, and will thus be negligible in ordinary vision. This is a distinctly fortunate fact, as the observation and measurement of the defect in living eyes would offer great difficulty, and its correction could only be effected by a much more clumsy optical appliance than an ordinary spectacle lens. On the other hand it is by no means certain that this small amount of angular distortion is without influence in refined observations. For example it may prevent some observers from attaining the extremely high accuracy reached by others in settings of the coincidence range-finder type, in which movement of the eyeball in its socket may be involved. In this connexion it is not unimportant to observe that an eye may be "skew" though self-corrected for astigmatism.

DISCUSSION

Mr A. Whitwell (communicated): As I am not familiar with the use of the eikonal I find it difficult to understand the method Mr Smith uses to correct toric lenses.

In the early part of the paper Mr Smith says "Consider for instance an expression which measures the time light takes to travel from a point of a fixed plane in the object space to a point of the exit pupil in the image space" and, later on, "The eikonal for the complete system is obtained by adding together those of its parts and giving the intermediate variables M', N' the values which make the sum stationary." Am I right in concluding that Mr Smith considers that when a small pencil of light proceeding from an object point arrives at a point on the exit pupil in the same phase the lens is satisfactorily corrected?

To explain my difficulty more clearly let me describe briefly the method usually adopted in designing a spherical spectacle lens for correcting distant vision. The conjugate point of the axial point of the fovea with respect to the optical system of the unaccommodated eye is called the far point. If now a correcting lens is placed so that its axis coincides with the optical axis of the eye and its focus with the far point, then the eye is generally considered to be corrected. A spherical lens, having two surfaces, has two degrees of freedom and it is possible to make an infinite number of lenses with the same axial power. It is usually considered that for all practical purposes any one of these lenses is as good as any other for correcting for points on the axis. When the eye moves on its centre of rotation its far point of course moves with it and the locus of the far point is called the far point sphere. At this stage it is usual to consider the eye as absent and to suppose that the chief

rays of oblique pencils emerging from the lens pass through the centre of rotation of the eye. The remaining degree of freedom at our disposal is now used to make the section of an oblique pencil by the far point sphere circular, assuming, naturally, that the exit pupil of the eye is circular.

In treating toric lenses it is usual to assume that the excursions of the eye take place only in the two meridians of maximum and minimum power. There are then two far point spheres, one corresponding to each of the two axial powers of the lens. The sections of an axial pencil emerging from the lens by the far point spheres are now elements of lines at right angles and consequently the sections of oblique pencils by the far point spheres should be similar elements of lines if oblique vision is to be as good as axial vision. To bring these four line elements on to the two spheres we need four extra degrees of freedom and in a sphero-toric lens we have only one at our disposal. Double toric lenses are not made. With our one degree of freedom it is possible to make the difference of the two oblique powers in one meridian equal to the difference of those in the other meridian. In other words it is possible to make the oblique astigmatism in the two meridians the same though not equal to the axial astigmatism. I have done this in a paper in the *Optician* of 8th August, 1924.

To return to spherical lenses, it is possible to use the remaining degree of freedom to make the circle of least confusion of oblique pencils of small obliquity lie on the far point sphere (*Optician*, 17th June, 1921). If now in addition we can make light from an object point arrive at the circle of least confusion in the same phase I agree that the image would be improved.

Mr Smith seems to me to lay more stress on getting the light reaching the retina from an object point in the same phase than on getting the image patch circular and focussed on the retina. It is possible that the one thing implies the other; if so I must put my difficulty down to my ignorance.

In conclusion I may say that in a paper in the *Optician* of 17th June, 1921, I calculated that a thin + 4 D. lens would have the circle of least confusion of an oblique pencil (say for an obliquity of 10°) on the far point sphere when the first surface was + 10·25 and the second − 6·25 D. The index of the glass was 1·53; the distance of the centre of rotation of the eye from the lens was 25 mm.

Might I ask Mr Smith to calculate the power of the two surfaces by his method so that I could compare the lenses say by trigonometrical computation?

Mr Smith (communicated): It will only be possible for me to give an answer to one or two of the interesting points Mr Whitwell raises as several of them relate to more advanced properties than those dealt with in the present paper.

The passage from which Mr Whitwell's first quotation is taken is only concerned with showing the basis on which the number of coefficients given in Table I rests. If it is once admitted that the optical properties of an instrument are determined by the length of the optical path, and that the mathematical expression representing this path length can be expanded in ascending powers of the variables mentioned, these numbers are found irrespective of particular theories of image formation.

The view adopted in the paper is that a satisfactory image is formed where the whole of the emergent wave is in the same phase. The coefficients calculated in the paper all correspond to the Gaussian coefficients of axially symmetrical systems. Assuming that aberrations are present in the system this investigation does not entitle us to say anything about decidedly oblique vision or on the best form of lens. I hope to discuss some of the problems relating to oblique vision in a future paper. It should however be noted that as a preliminary to such a discussion the 35 coefficients mentioned in Group 2 will require investigation.

I am much interested in Mr Whitwell's account of the method used in calculating best form lenses. If the validity of deductions from the cross-section of pencils of rays were admitted, the method would have much to commend it. I imagine the real position is that this validity has been assumed; it would be altogether more satisfactory to examine it on the basis of the wave theory, and definitely either justify it or reject it. I hope that the future work I have referred to will include such an examination of this procedure. I will bear in mind Mr Whitwell's suggestion that actual lenses should be calculated. I ought however to point out that should the analytical enquiry lead to a conclusion unfavourable to the method of investigation now adopted, trigonometrical computation in its ordinary significance will have been ruled out of court as a criterion for comparing forms derived on distinct theories of image formation. Such calculations would have to be used as a basis for the accurate computation of differences of path at the assumed image point.

CANONICAL FORMS IN THE THEORY OF ASYMMETRICAL OPTICAL SYSTEMS

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ABSTRACT. Canonical forms for the quadratic terms of the eikonal and of the characteristic function for any optical instrument are found to involve only six arbitrary constants. In the eikonal defined by

$\mathscr{E}$ = constant + first order terms

$+ \frac{1}{2}aM^2 + \frac{1}{2}bN^2 + \frac{1}{2}cM'^2 + \frac{1}{2}dN'^2 + eMN + fM'N' - gMN' - hM'N - iMM' - jNN'$

+ higher order terms,

in addition to removing the first order terms we may take any of the following simplifying conditions:

$$a = b = c = d = 0,$$

$$a + b = c + d = e = f = 0,$$

$$a + b = c + d = g = h = 0.$$

Similarly in the characteristic function

V = constant + first order terms

$+ \frac{1}{2}Ay^2 + \frac{1}{2}Bz^2 + \frac{1}{2}Cy'^2 + \frac{1}{2}Dz'^2 + Eyz + Fy'z' - Gyz' - Hy'z - Iyy' - Jzz'$

+ higher order terms,

the first order terms may be removed and in addition we may put

$$A = B = C = D = 0,$$

or

$$A + B = C + D = E = F = 0,$$

or

$$A + B = C + D = G = H = 0.$$

The seven constant canonical form of the characteristic function obtained by Larmor, viz.

V = constant $+ \frac{1}{2}Ay^2 + \frac{1}{2}Bz^2 + \frac{1}{2}Cy'^2 + \frac{1}{2}Dz'^2 + Eyz + Fy'z' - I(yy' + zz') + \ldots,$

is not general. Larmor's theorem on the equivalence of any optical system to a symmetrical instrument together with two thin astigmatic lenses also fails. Three separated astigmatic lenses are needed to represent the general system.

Introductory

For many purposes it is important to reduce to as simple a form as possible any mathematical function. This simplification often takes the form of reducing both the number of independent constants and the number of terms present in the function. When the function represents in a mathematical form the properties of a class of physical instruments, the simplified expression will also represent all the

essential properties though some of the usual conventions may no longer hold. Thus Larmor* has obtained a form for the second order terms of the characteristic function of an asymmetrical optical system which contains only seven arbitrary constants instead of the ten constants present in the most general expression of this order. This implies that for mathematical investigations which are only concerned with the essential correspondencies subsisting between the two spaces on either side of such a system, only seven degrees of freedom need be considered. It does not imply that in an actual instrument we may not have to take into account a greater number of variables. For example, it is quite immaterial in theory if the image of a vertical object is horizontal; in an actual instrument such an effect would be of first rate importance. In practical applications, particularly those involving numerical work, we therefore expect to deal with a greater number of constants than need be introduced into some theoretical discussions. It may therefore happen that the simplification of the expressions will only prove of importance because it facilitates the discussion of points in the general theory of the instrument.

We proceed to show that the simplest expressions governing the properties of the general asymmetrical optical system contain only six constants in place of the seven found by Larmor.

Canonical Forms of the Eikonal

The simplest demonstration that six constants suffice is perhaps obtained by the use of the eikonal. Whatever reference points and axes may be chosen, this function involves all the properties of the instrument. By choosing particular reference points on a selected ray, and suitably fixing the directions of the axes of reference, a number of constants may be deleted.

The direction of the ray will be specified in the object space by its direction cosines (L, M, N), of which only M and N can be regarded as independent variables. In the image space the corresponding accented letters will be used. The axes of x and x', not necessarily coincident, can obviously be chosen to remove from the eikonal any terms linear in M, N, M', N'. The most general expression of the next order is then

$$\mathcal{E} = \tfrac{1}{2}aM^2 + \tfrac{1}{2}bN^2 + \tfrac{1}{2}cM'^2 + \tfrac{1}{2}dN'^2 + eMN + fM'N' - gMN' - hM'N - iMM' - jNN'.$$

By moving the reference points to x, x', we add the terms

$$\tfrac{1}{2}\mu x\,(M^2 + N^2) - \tfrac{1}{2}\mu' x'\,(M'^2 + N'^2);$$

that is to say, if we regard a, b, c, d as combinations of the four variables $\phi, \psi, \gamma, \lambda$ where

$$\phi = \tfrac{1}{2}(a + b),\ \psi = \tfrac{1}{2}(c + d),\ \gamma = \tfrac{1}{2}(a - b),\ \lambda = \tfrac{1}{2}(c - d),$$

the only two coefficients which we can alter in value by a change of origin are ϕ and ψ.

We now consider the effect of rotating the y and z axes about the axis of x, and similarly the y' and z' axes about the axis of x'. It is quite simple by the direct

* *Proc. Lond. Math. Soc.* **20** 1889 (181).

substitutions $M\cos\alpha - N\sin\alpha$, $M\sin\alpha + N\cos\alpha$, $M'\cos\beta - N'\sin\beta$, $M'\sin\beta + N'\cos\beta$ for M, N, M', N' respectively to show that we have a number of invariants for any rotation, such as

$$\phi,\ \psi \qquad \ldots.(1),$$

$$\gamma^2 + e^2,\ \lambda^2 + f^2,\ g^2 + h^2 + i^2 + j^2,\ ij - gh \qquad \ldots.(2),$$

$$\gamma(g^2 - h^2 + i^2 - j^2) + 2e(gj + hi),\ 2\gamma(gj + hi) - e(g^2 - h^2 + i^2 - j^2) \qquad \ldots.(3),$$

and others. Of these quantities eight, including ϕ and ψ, are independent. The most convenient form for our purpose in which to express these is to note that the rotations of the axes are equivalent to substituting the matrix product

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \phi+\gamma & e & i & g \\ e & \phi-\gamma & h & j \\ i & h & \psi+\lambda & f \\ g & j & f & \psi-\lambda \end{bmatrix}$$

$$\times \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \\ 0 & 0 & -\sin\beta & \cos\beta \end{bmatrix}$$

for the central matrix alone. Since the determinants associated with the two outer matrices are each equal to unity, and since ϕ and ψ are arbitrary quantities which are themselves unaffected by the axial rotations, the coefficients of $\phi^2\psi^2$, ϕ^2, ψ^2, $\phi\psi$, ϕ, ψ, and the term without ϕ or ψ, in the determinant

$$\begin{vmatrix} \phi+\gamma & e & i & g \\ e & \phi-\gamma & h & j \\ i & h & \psi+\lambda & f \\ g & j & f & \psi-\lambda \end{vmatrix} = \Delta \qquad \ldots.(4),$$

as well as Δ itself, must be unaltered by any rotation of the axes. If we write

$$\Delta = \phi^2\psi^2 - p\phi^2 - q\psi^2 - r\phi\psi + s\phi + t\psi + u \qquad \ldots.(5)$$

the invariants are given by

$$p = \lambda^2 + f^2,\ q = \gamma^2 + e^2,\ r = g^2 + h^2 + i^2 + j^2,$$

s = sum of minors of a and b, i.e. $\phi+\gamma$ and $\phi-\gamma$,
t = sum of minors of c and d, i.e. $\psi+\lambda$ and $\psi-\lambda$,
u = value of Δ
} when $\phi = \psi = 0$.

It is obvious that these invariants are all independent. It follows that there cannot be less than six independent constants in any formula relating to general systems. On the other hand there need not be more than six. We may choose the origins to make $\phi = \psi = 0$, and rotate the axes to satisfy two additional conditions, such as $\gamma = \lambda = 0$, or $e = f = 0$, or $g = h = 0$. The new values of the other variables corresponding to these simplified forms of the eikonal are easily found from the invariants. Where there is any ambiguity of sign it is immaterial which sign is taken.

Canonical Forms of the Characteristic Function

Since the whole of the properties of any asymmetrical system are involved in the values of six invariants, it is obvious that only six independent coefficients are necessary in the characteristic function. If we denote this function by

V = constant + first order terms

$+ \frac{1}{2}Ay^2 + \frac{1}{2}Bz^2 + \frac{1}{2}Cy'^2 + \frac{1}{2}Dz'^2 + Eyz + Fy'z' - Gyz' - Hy'z - Iyy' - Jzz'$

+ higher order terms

we can remove the first order terms in the usual way, and, since the effect of rotation of the axes is represented by the same substitutions when y, z, y', z' are the variables as when M, N, M', N' are employed, we obtain corresponding invariants on rotation.

One of the chief differences between the characteristic function and the eikonal is the comparative simplicity of the changes in the coefficients of the latter function due to changes of the two origins. The alterations in the coefficients of the characteristic function may be found, as Maxwell and Larmor have done, from first principles, but it is simpler to make use of the relation between the characteristic function and the eikonal.

The definitions of these functions in terms of optical paths show that they are connected by the equation

$$V = \mathscr{E} + \mu'M'y' + \mu'N'z' - \mu My - \mu Nz \qquad \ldots.(6),$$

where (y, z) and (y', z') are the points of intersection of the ray with the planes $x = 0$ and $x' = 0$. From the fundamental property of the eikonal it follows that

$$\frac{\partial V}{\partial M} = \frac{\partial V}{\partial N} = \frac{\partial V}{\partial M'} = \frac{\partial V}{\partial N'} = 0 \qquad \ldots.(7)$$

and these conditions enable M, N, M', N' to be eliminated from (6). Since for the purposes of this discussion we are only concerned with the terms of the second degree in $\mathscr{E}$, we have

$$V = \tfrac{1}{2}(\mu'M'y' + \mu'N'z' - \mu My - \mu Nz) \qquad \ldots.(8),$$

where M, N, M', N' receive the values which satisfy (7). From (6), (7), and (8) we then have, in terms of the coefficients of $\mathscr{E}$,

$$\begin{vmatrix} -2V & y & z & y' & z' \\ y & a & e & i & g \\ z & e & b & h & j \\ y' & i & h & c & f \\ z' & g & j & f & d \end{vmatrix} = 0 \qquad \ldots.(9).$$

If now only six of the ten coefficients $a, \ldots, j$ are present, we have obtained a form of V involving only six coefficients. On the other hand there will, in general, be

present all the possible terms in y, z, y', z'. For instance if $a = b = c = d = 0$, we find

$$\begin{aligned} V\,(ef - gh - ij)^2 &+ fhjy^2 + fgiz^2 + egjy'^2 + ehiz'^2 \\ &+ (fyz + ey'z')\,(ef - gh - ij) \\ &- (hyz' + gy'z)\,(ef - gh + ij) \\ &- (jyy' + izz')\,(ef + gh - ij) = 0, \end{aligned}$$

and this is clearly not the type of expression we are seeking.

From the comparison of (9) with the initial formula for V, we see that the capitals $A, B, \ldots, \mathcal{J}$ denote the minors of $a, b, \ldots, j$ in Δ divided by $-\Delta$. This enables us to write down equations connecting the invariants on rotation of the two functions. Denoting the invariants of the characteristic function by the letters P, Q, R, S, T, U, into which A, B, C, D always enter in the forms $(A - B)$ and $(C - D)$, and also writing w and W for the two invariants $ij - gh$ and $I\mathcal{J} - GH$, which do not enter directly into the system we have adopted, we obtain the relations

$$\left.\begin{aligned} -2\Delta\Phi &= s - 2p\phi - r\psi + 2\phi\psi^2 \\ -2\Delta\Psi &= t - 2q\psi - r\phi + 2\psi\phi^2 \\ \Delta W &= w \\ \Delta\,(P - \Psi^2) &= q - \phi^2 \\ \Delta\,(Q - \Phi^2) &= p - \psi^2 \\ \Delta\,(R - 4\Phi\Psi) &= r - 4\phi\psi \\ -\Delta\,(S - 2P\Phi - R\Psi + 2\Phi\Psi^2) &= 2\phi \\ -\Delta\,(T - 2Q\Psi - R\Phi + 2\Psi\Phi^2) &= 2\psi \\ \Delta\,(U + S\Phi + T\Psi - P\Phi^2 - Q\Psi^2 - R\Phi\Psi + \Phi^2\Psi^2) &= 1 \end{aligned}\right\} \ldots.(10),$$

the quantity in the last bracket being the determinant formed from the capital letters corresponding to Δ for the small letters. The Greek capitals Φ and Ψ denote, of course, $\frac{1}{2}(A + B)$ and $\frac{1}{2}(C + D)$.

We will now proceed to obtain canonical forms similar to those established for the eikonal. The initial step is to secure zero values for Φ and Ψ. The first pair of equations (10) shows, since s and t are absolute invariants of the system, that the values of ϕ and ψ will in general be finite, that is to say the origins for the canonical forms of the characteristic function will differ from those found necessary for the eikonal. We may not use the last three of equations (10) to obtain a solution, for the invariants represented by capital letters change in value as the origins are moved. It is therefore necessary to solve the first two equations. Let us write ξ for $\phi\psi - \frac{1}{2}r$. Then

$$\frac{1}{pq - \xi^2} = \frac{2\phi}{sq + t\xi} = \frac{2\psi}{tp + s\xi} \qquad \ldots.(11)$$

so that

$$2\,(r + 2\xi)\,(pq - \xi^2)^2 - (sq + t\xi)\,(tp + s\xi) = 0.$$

This equation is of the fifth order, and therefore always has a real root. When it has been obtained ϕ and ψ are given without ambiguity by (11). It only remains

to move the two origins through the distances ϕ and ψ, and then rotate the axes suitably, to secure the general expression of the characteristic function in one of the forms having

$$A = B = C = D = 0,$$

or $$A + B = C + D = E = F = 0,$$

or $$A + B = C + D = G = H = 0.$$

Larmor's Theory

It is of special importance to discuss the seven constant form put forward by Larmor, because from it he has derived the equivalence of the general system to an artificial system consisting of two thin astigmatic lenses separated by a symmetrical system. This combination may, however, be simplified, for any symmetrical system is equivalent to two thin lenses in distinct but otherwise arbitrary positions. By placing these equivalent lenses in the same positions as the astigmatic lenses, Larmor's conclusion may be stated in the form that any optical system is equivalent to two thin astigmatic lenses occupying special positions. If this conclusion were correct, we should have to admit that the number of independent constants could always be less than six. For each single lens has three independent coefficients, and the only other element of the system is the separation of these lenses. This total of seven constants will be reduced to at most four by the correct choice of the two origins and the two rotations of the axes.

Larmor's second order terms are

$$V = \tfrac{1}{2}Ay^2 + \tfrac{1}{2}Bz^2 + \tfrac{1}{2}Cy'^2 + \tfrac{1}{2}Dz'^2 + Eyz + Fy'z' - I(yy' + zz').$$

The seven constants can obviously be reduced to six: for any rotation of the axes will lead to the substitution of $I(yy' + zz') + G(yz' - y'z)$ for $I(yy' + zz')$, an increase of one in the number of coefficients. Each of the two rotations may now be chosen to remove one of the remaining coefficients: for instance we may secure on the one side either $A - B = 0$ or $E = 0$, and on the other side $C - D = 0$ or $F = 0$. The appearance of more distinct coefficients than are essential is, from the present point of view, a defect in the expression.

Of greater moment is the fact that the values of the invariants are not all arbitrary. Since W does not belong to our fundamental system of invariants we may pass by the relation $R = 2W$, but we cannot regard $S = T = 0$, which follow at once from the definitions, so lightly. They appear in fact to be fatal to Larmor's claim of generality for his formula. The only line of defence which can be taken rests upon the fact that the invariants which we have denoted by capital letters are not absolute invariants of the system. In this they differ from the invariants of the eikonal.

The author has attempted to demonstrate this want of generality using only the relations (5) and (10), but these proofs appear to be inconclusive owing to the difficulty of excluding certain special suppositions. The following discussion makes

use of an expression for the characteristic function in which the coefficients are explicit functions of the quantities which it is most convenient to calculate in the numerical computation of asymmetrical systems. The definitions of these quantities and the formulae by which they may be computed have been given elsewhere in these *Transactions**.

From (9), in the new notation, we immediately find

$$\begin{vmatrix} -2V & y & z & y' & z' \\ \mathcal{J}y - \mathcal{G}z & \mathcal{A} & -\mathcal{G} & 1 & 0 \\ \mathcal{G}z - \mathcal{H}y & -\mathcal{I} & \mathcal{B} & 0 & 1 \\ \mathcal{J}y' - \mathcal{H}z' & 1 & 0 & \mathcal{C} & -\mathcal{U} \\ \mathcal{G}z' - \mathcal{G}y' & 0 & 1 & -\mathcal{V} & \mathcal{D} \end{vmatrix} = 0 \quad \ldots.(12),$$

where the coefficients in the first column are not affected by changes of origin. Between the twelve coefficients there subsist two identical relations

$$\mathcal{A}\mathcal{H} - \mathcal{I}\mathcal{J} \equiv \mathcal{B}\mathcal{G} - \mathcal{G}\mathcal{I}, \quad \mathcal{C}\mathcal{G} - \mathcal{V}\mathcal{J} \equiv \mathcal{D}\mathcal{H} - \mathcal{U}\mathcal{I} \qquad \ldots.(13).$$

The coefficient of $-2V$ is

$$(\mathcal{A}\mathcal{B} - \mathcal{G}\mathcal{I})(\mathcal{C}\mathcal{D} - \mathcal{U}\mathcal{V}) - \mathcal{A}\mathcal{C} - \mathcal{B}\mathcal{D} - \mathcal{G}\mathcal{V} - \mathcal{I}\mathcal{U} + 1 = \Delta \quad \ldots.(14),$$

and the determinant adjugate to Δ is

$$\begin{vmatrix} \mathcal{B}(\mathcal{C}\mathcal{D} - \mathcal{U}\mathcal{V}) - \mathcal{C} & \mathcal{I}(\mathcal{C}\mathcal{D} - \mathcal{U}\mathcal{V}) + \mathcal{V} & 1 - \mathcal{B}\mathcal{D} - \mathcal{I}\mathcal{U} & -\mathcal{B}\mathcal{V} - \mathcal{C}\mathcal{I} \\ \mathcal{G}(\mathcal{C}\mathcal{D} - \mathcal{U}\mathcal{V}) + \mathcal{U} & \mathcal{A}(\mathcal{C}\mathcal{D} - \mathcal{U}\mathcal{V}) - \mathcal{D} & -\mathcal{A}\mathcal{U} - \mathcal{D}\mathcal{G} & 1 - \mathcal{A}\mathcal{C} - \mathcal{G}\mathcal{V} \\ 1 - \mathcal{B}\mathcal{D} - \mathcal{G}\mathcal{V} & -\mathcal{A}\mathcal{V} - \mathcal{D}\mathcal{I} & \mathcal{D}(\mathcal{A}\mathcal{B} - \mathcal{G}\mathcal{I}) - \mathcal{A} & \mathcal{V}(\mathcal{A}\mathcal{B} - \mathcal{G}\mathcal{I}) + \mathcal{I} \\ -\mathcal{B}\mathcal{U} - \mathcal{C}\mathcal{G} & 1 - \mathcal{A}\mathcal{C} - \mathcal{I}\mathcal{U} & \mathcal{U}(\mathcal{A}\mathcal{B} - \mathcal{G}\mathcal{I}) + \mathcal{G} & \mathcal{C}(\mathcal{A}\mathcal{B} - \mathcal{G}\mathcal{I}) - \mathcal{B} \end{vmatrix}$$
$$\ldots.(15).$$

From (12) we note that only two columns yield any contribution to the coefficient of any particular mixed term, such as yz or yy'. The identities (13) show that the total contributions from these two columns are equal. We can thus write down in two distinct forms the conditions that definite relations shall hold between particular coefficients. To investigate Larmor's theory we must suppose that the origins in the object and image spaces may be chosen to make the coefficients of yy' and zz' each equal to $-\lambda/\Delta$, and the coefficients of yz' and $-y'z$ each equal to $-\mu/\Delta$. These conditions require the following relations to hold between two of the elements of each of the four columns

$$\frac{1 - \mathcal{B}\mathcal{D} - \mathcal{G}\mathcal{V}}{\mathcal{G}\lambda + \mathcal{J}\mu} = -\frac{\mathcal{B}\mathcal{U} + \mathcal{C}\mathcal{G}}{\mathcal{J}\mu + \mathcal{H}\lambda} = \frac{1 - \mathcal{A}\mathcal{C} - \mathcal{I}\mathcal{U}}{\mathcal{J}\lambda - \mathcal{H}\mu} = \frac{\mathcal{A}\mathcal{V} + \mathcal{D}\mathcal{I}}{\mathcal{J}\mu - \mathcal{G}\lambda} = \frac{1 - \mathcal{B}\mathcal{D} - \mathcal{I}\mathcal{U}}{\mathcal{G}\lambda - \mathcal{H}\mu}$$
$$= \frac{\mathcal{A}\mathcal{U} + \mathcal{D}\mathcal{G}}{\mathcal{J}\mu - \mathcal{G}\lambda} = \frac{1 - \mathcal{A}\mathcal{C} - \mathcal{G}\mathcal{V}}{\mathcal{J}\lambda + \mathcal{G}\mu} = -\frac{\mathcal{B}\mathcal{V} + \mathcal{C}\mathcal{I}}{\mathcal{G}\mu + \mathcal{H}\lambda} = \frac{1}{\mathcal{J}\mathcal{J} - \mathcal{G}\mathcal{H}}$$
$$\ldots.(16).$$

* "On Toric Lenses," *Trans. Opt. Soc.* **29** 1927–8 (71).

On substituting these values in (15) we obtain an equation for V which may be written in the form

$2V\Delta =$

	y	z	y'	z'
y	$\mathscr{C}\lambda - \mathscr{U}\mu$	$-\mathscr{U}\lambda - \mathscr{C}\mu$	$-\lambda$	$-\mu$
z	$-\mathscr{V}\lambda + \mathscr{D}\mu$	$\mathscr{D}\lambda + \mathscr{V}\mu$	μ	$-\lambda$
y'	$-\lambda$	μ	$\mathscr{A}\lambda + \mathscr{G}\mu$	$-\mathscr{G}\lambda + \mathscr{A}\mu$
z'	$-\mu$	$-\lambda$	$-\mathscr{J}\lambda - \mathscr{B}\mu$	$\mathscr{B}\lambda - \mathscr{J}\mu$

where the original columnar arrangement is retained. From the comparison of the two contributions to yz and $y'z'$ it follows that in any system capable of representation by Larmor's formula it must be possible to satisfy, in addition to other conditions, the condition

$$\frac{\mathscr{A}+\mathscr{B}}{\mathscr{G}-\mathscr{J}} = \frac{\mathscr{C}+\mathscr{D}}{\mathscr{V}-\mathscr{U}} \qquad \ldots.(17).$$

Now if the object and image space origins are moved away from the lenses through reduced distances τ and τ' respectively, the new values of these eight coefficients are

$$\mathscr{A} - \mathscr{J}\tau, \quad \mathscr{B} - \mathscr{G}\tau, \quad \mathscr{G} - \mathscr{B}\tau, \quad \mathscr{J} - \mathscr{A}\tau,$$
$$\mathscr{C} - \mathscr{J}\tau', \quad \mathscr{D} - \mathscr{G}\tau', \quad \mathscr{V} - \mathscr{B}\tau', \quad \mathscr{U} - \mathscr{A}\tau'.$$

If then we have an optical system in which, with any origins, the two fractions $(\mathscr{A}+\mathscr{B})/(\mathscr{G}-\mathscr{J})$ and $(\mathscr{C}+\mathscr{D})/(\mathscr{V}-\mathscr{U})$ are unequal, but one of them is equal to $(\mathscr{I}+\mathscr{J})/(\mathscr{G}-\mathscr{H})$, this fraction will be unaltered in value by any change of the origin, and the other cannot be made equal to it for any finite movement of the other origin. Such systems cannot be represented by Larmor's formula.

Another special system is one in which for some positions of the origins it is found that

$$\frac{\mathscr{A}+\mathscr{B}}{\mathscr{G}-\mathscr{J}} = \frac{\mathscr{C}+\mathscr{D}}{\mathscr{V}-\mathscr{U}} = \frac{\mathscr{I}+\mathscr{J}}{\mathscr{G}-\mathscr{H}}.$$

No change of origin produces any alteration in the value of any of these fractions; they may therefore be chosen to satisfy other conditions. If they are chosen to make $\mathscr{A}+\mathscr{B}$ and $\mathscr{C}+\mathscr{D}$ both vanish, the same values of τ and τ' will cause $\mathscr{G}-\mathscr{J}$ and $\mathscr{V}-\mathscr{U}$ to vanish also. Equations (16) then show that we must also have

$$\mathscr{I} - \mathscr{J} = \mathscr{G} + \mathscr{H} = 0, \quad \text{or} \quad \lambda = \mu = 0,$$

or either $\lambda = 0$ or $\mu = 0$ together with other conditions.

The simultaneous disappearance of λ and μ must be ruled out as of no account. If we combine the products of the first and third and of the second and fourth of the equal fractions in (16), we find that the square of any one of them is equal to $\Delta/(\lambda^2+\mu^2)\,(\mathscr{J}\mathscr{J} - \mathscr{H}\mathscr{H})$, or, by the last fraction, each fraction in (16) is equal to $\Delta/(\lambda^2+\mu^2)$. The vanishing of both λ and μ thus involves the vanishing of Δ, which implies a selection of origins for which the characteristic function does not exist. The remaining alternatives reduce to the first one. In fact if we take the sum of the squares of the first four numerators in equations (16) we obtain

$$2\Delta + (\mathscr{A}\mathscr{C} - \mathscr{B}\mathscr{D} - \mathscr{G}\mathscr{V} + \mathscr{J}\mathscr{U})^2 + (\mathscr{A}\mathscr{V} + \mathscr{B}\mathscr{U} + \mathscr{C}\mathscr{G} + \mathscr{D}\mathscr{J})^2 \quad (18),$$

and the conditions

$$\mathscr{A}+\mathscr{B}=\mathscr{C}+\mathscr{D}=\mathscr{S}-\mathscr{T}=\mathscr{U}-\mathscr{V}=0$$

involve the disappearance of the quantities in both the brackets of (18). Since the sum of the squares of the first four denominators in (16) is

$$(\lambda^2+\mu^2)(\mathscr{I}^2+\mathscr{J}^2+\mathscr{S}^2+\mathscr{K}^2)$$

it is evident that the systems we are considering satisfy

$$\mathscr{I}^2+\mathscr{J}^2+\mathscr{S}^2+\mathscr{K}^2=2(\mathscr{I}\mathscr{J}-\mathscr{S}\mathscr{K})$$

or

$$\mathscr{I}-\mathscr{J}=\mathscr{S}+\mathscr{K}=0.$$

It at once follows that these systems are not general.

We now only have to consider systems in which $(\mathscr{I}+\mathscr{J})/(\mathscr{S}-\mathscr{K})$ differs from both the fractions given in (17). This implies that we can satisfy (17) for any arbitrarily given value of the ratio λ/μ. Let us choose the origins to make both the numerators in (17) vanish. From (16) we then have either $\lambda=0$ or $\mathscr{S}+\mathscr{K}=0$ and $\mathscr{I}-\mathscr{J}=0$. If the alternative $\lambda=0$ is taken, we note that from the first four of equations (16) each fraction is equal to both

$$\frac{\mathscr{A}\mathscr{V}+\mathscr{B}\mathscr{U}+\mathscr{C}\mathscr{S}+\mathscr{D}\mathscr{T}}{(\mathscr{I}-\mathscr{J})\mu-(\mathscr{S}+\mathscr{K})\lambda} \text{ and } \frac{\mathscr{A}\mathscr{C}-\mathscr{B}\mathscr{D}-\mathscr{S}\mathscr{V}+\mathscr{T}\mathscr{U}}{(\mathscr{I}-\mathscr{J})\lambda+(\mathscr{S}+\mathscr{K})\mu}.$$

Transforming by (13) the product of the numerator of the former and the denominator of the latter, we obtain

$$\lambda[(\mathscr{A}\mathscr{I}+\mathscr{B}\mathscr{J})(\mathscr{U}-\mathscr{V})+(\mathscr{C}\mathscr{J}+\mathscr{D}\mathscr{I})(\mathscr{S}-\mathscr{T})+(\mathscr{S}\mathscr{V}-\mathscr{T}\mathscr{U})(\mathscr{S}+\mathscr{K})]$$
$$=\mu[(\mathscr{U}\mathscr{K}+\mathscr{V}\mathscr{S})(\mathscr{A}+\mathscr{B})+(\mathscr{S}\mathscr{K}+\mathscr{T}\mathscr{S})(\mathscr{C}+\mathscr{D})+(\mathscr{S}\mathscr{V}-\mathscr{T}\mathscr{U})(\mathscr{I}-\mathscr{J})].$$

Thus in the case we are considering, viz. $\lambda=\mathscr{A}+\mathscr{B}=\mathscr{C}+\mathscr{D}=0$, $\mu\neq 0$, we have either $\mathscr{I}=\mathscr{J}$ or $\mathscr{S}\mathscr{V}=\mathscr{T}\mathscr{U}$. The latter by (16) involves $\mathscr{S}+\mathscr{K}=0$. By considering only a single position of the origins, viz. one depending on the original accidental orientation of the axes, we have now reached the conclusion that at least one of the conditions $\mathscr{I}-\mathscr{J}=0$, $\mathscr{S}+\mathscr{K}=0$, must be satisfied whenever Larmor's formula is applicable. By rotating the system as a whole through any angle it is immediately seen that both these conditions must always be satisfied.

It is unnecessary to go farther and determine how many degrees of freedom such systems may possess. We will merely note here that the characteristic function for a single asymmetrical refracting surface cannot conform to Larmor's type, for the conditions $\mathscr{I}-\mathscr{J}=\mathscr{S}+\mathscr{K}=0$ reduce it at once to the expression typical of symmetrical systems. As a corollary to this failure, it follows that the theorem on the equivalence of the general system to the special system devised by Larmor, which stands or falls with the legitimacy of the relations $\mathscr{I}-\mathscr{J}=\mathscr{S}+\mathscr{K}=0$, is incorrect.

To construct a general system we require in the first place to possess ten degrees of freedom, and this involves at least ten independent constructional elements.

Three separated surfaces yield a total of eleven arbitrary quantities, and we conclude that no system having less than three separated refracting surfaces may be adopted as a general pattern.

Thin Systems

As in all other branches of geometrical optics, it is of importance to consider the forms characteristic of thin systems of lenses. The usual equations for the calculation of the coefficients become

$$\mathscr{A} = \mathscr{B} = \mathscr{C} = \mathscr{D} = 1$$

$$\mathscr{G} = \mathscr{I} = \mathscr{U} = \mathscr{V} = 0$$

and $\mathscr{G}$, $\mathscr{H}$, $\mathscr{I}$, $\mathscr{J}$ are the simple sums of the corresponding quantities for the component surfaces. It follows that $\mathscr{G} = \mathscr{H}$ for thin systems, and by rotation of the axes we may secure $\mathscr{G} = \mathscr{H} = 0$. In constructing the eikonal we may take the point in which the refracting surfaces meet the axis as the origin for both spaces, and obtain in the general case

$$2\mathscr{E}(\mathscr{I}\mathscr{J} - \mathscr{G}^2) = \mathscr{I}(M' - M)^2 + 2\mathscr{G}(M' - M)(N' - N) + \mathscr{J}(N' - N)^2,$$

and the two constant simplified form

$$\mathscr{E} = \frac{(M' - M)^2}{2\mathscr{J}} + \frac{(N' - N)^2}{2\mathscr{I}}$$

when the axes are rotated to make $\mathscr{G} = 0$. The important feature of the last expression is that the M's and N's occur in separate terms.

Since these origins are conjugate to one another there is no corresponding expression for the characteristic function. If we take origins in quite general positions[9] we can easily obtain the general formula for V. The most compact expression for V is

$$\begin{aligned}[2V\tau\tau' - (y^2 + z^2)\tau' - (y'^2 + z'^2)\tau][(\tau + \tau')^2 - (\mathscr{I} + \mathscr{J})(\tau + \tau')\tau\tau' + (\mathscr{I}\mathscr{J} - \mathscr{G}^2)\tau^2\tau'^2] \\ + (y\tau' + y'\tau)^2(\tau + \tau' - \mathscr{I}\tau\tau') + (z\tau' + z'\tau)^2(\tau + \tau' - \mathscr{J}\tau\tau') \\ - 2(y\tau' + y'\tau)(z\tau' + z'\tau)\mathscr{G}\tau\tau' = 0.\end{aligned}$$

From this expression it is evident that the most interesting cases are obtained by selecting them at equal optical distances on either side of the lens. If this distance is ξ, and if we assume $\mathscr{G} = 0$, we obtain

$$2V\xi = \frac{(y^2 + y'^2)(1 - \xi\mathscr{J}) - 2yy'}{2 - \xi\mathscr{J}} + \frac{(z^2 + z'^2)(1 - \xi\mathscr{I}) - 2zz'}{2 - \xi\mathscr{I}}$$

where it will be noted that the y's and z's are separated. Now putting

$$\xi = \frac{2}{\mathscr{I} + \mathscr{J}},$$

we obtain

$$V = \frac{\mathscr{F}+\mathscr{f}}{8}\left[\frac{(y^2+y'^2)(\mathscr{F}-\mathscr{f})-2yy'(\mathscr{F}+\mathscr{f})}{\mathscr{F}} - \frac{(z^2+z'^2)(\mathscr{F}-\mathscr{f})-2zz'(\mathscr{F}+\mathscr{f})}{\mathscr{f}}\right],$$

which involves as constants $\mathscr{F}+\mathscr{f}$ and $\mathscr{F}/\mathscr{f}$. The corresponding form of eikonal is

$$\mathscr{E} = \frac{1}{2(\mathscr{F}+\mathscr{f})}\left[\frac{(M^2+M'^2)(\mathscr{F}-\mathscr{f})-2MM'(\mathscr{F}+\mathscr{f})}{\mathscr{F}} - \frac{(N^2+N'^2)(\mathscr{F}-\mathscr{f})+2NN'(\mathscr{F}+\mathscr{f})}{\mathscr{f}}\right].$$

Again if we put

$$\xi = \frac{\mathscr{F}+\mathscr{f}}{\mathscr{F}\mathscr{f}}$$

we find for the characteristic function

$$V = \frac{\mathscr{F}\mathscr{f}}{2(\mathscr{F}^2-\mathscr{f}^2)}[\mathscr{F}(z^2+z'^2-2yy')-\mathscr{f}(y^2+y'^2-2zz')],$$

and for the eikonal

$$\mathscr{E} = -\frac{1}{2\mathscr{F}}(M^2+M'^2+2NN') - \frac{1}{2\mathscr{f}}(N^2+N'^2+2MM').$$

To obtain the eikonal in its canonical forms we put

$$\xi = \frac{\mathscr{F}+\mathscr{f}}{2\mathscr{F}\mathscr{f}}$$

which leads to

$$\mathscr{E} = \tfrac{1}{4}[(M-M')^2-(N-N')^2](1/\mathscr{f}-1/\mathscr{F}) - \tfrac{1}{2}(MM'+NN')(1/\mathscr{F}+1/\mathscr{f}).$$

The remaining form is obtained by rotating both sets of axes through half a right angle.

The canonical forms of the characteristic function are obtained by giving ξ one of the values which satisfies the equation

$$2\mathscr{F}\mathscr{f}\xi^2 - 3(\mathscr{F}+\mathscr{f})\xi + 4 = 0,$$

that is

$$\frac{1-\xi\mathscr{f}}{2-\xi\mathscr{f}} = -\frac{1-\xi\mathscr{F}}{2-\xi\mathscr{F}} = \frac{(\mathscr{F}-\mathscr{f})}{2(\mathscr{F}+\mathscr{f}-\xi\mathscr{F}\mathscr{f})}$$

from which the formula

$$V = \frac{1}{\xi}\left[\frac{\{(y+y')^2-(z+z')^2\}(\mathscr{F}-\mathscr{f})}{4(\mathscr{F}+\mathscr{f}-\xi\mathscr{F}\mathscr{f})} - (yy'+zz')\right]$$

at once follows. The remaining canonical form is obtained as with the eikonal by rotating the axes through half a right angle. The solution of the equation may be written

$$4\xi\mathscr{F}\mathscr{f} = 3(\mathscr{F}+\mathscr{f}) \pm \{(\mathscr{F}+\mathscr{f})^2+8(\mathscr{F}-\mathscr{f})^2\}^{\frac{1}{2}}$$

so that the roots are always real.

REVIEWS

Encyclopédie Photométrique. Première section—Généralités. I. *Introduction Générale à la Photométrie.* By CHARLES FABRY. Pp. viii + 178; 50 Figs. (Paris: Revue d'Optique théorique et instrumentale, 1927.) Price 18 fr.

This book forms the introductory portion of the first volume of the *Encyclopédie Photométrique* which is in process of preparation under the editorship of Prof. Fabry. The encyclopaedia itself is on original lines as its subject matter is based on a very extended definition of photometry. One naturally regards photometry as the science dealing with visible radiations and the nearby ultra-violet and infra-red radiations which are studied by the same general instrumental methods (excluding the eye, of course). In this work photometry is interpreted to cover the study and measurement of the whole spectrum of radiations from γ rays to wireless waves. We should, however, have thought that the work would have been better entitled the *Encyclopédie Radiométrique*, which is more indicative of its fundamental character and ambitious scope.

This volume by Prof. Fabry forms the introduction to the whole work, and consists of an excellent exposition of the basic principles and theory of radiation treated as fully as the practical requirements of the subject matter of the succeeding sections necessitate. In consequence, it is mainly concerned with the properties of radiation and radiating bodies, and the special treatment of the visible radiations due to their psycho-physiological property of affecting a sense organ.

The first chapter deals generally with radiation and the complete spectrum of radiation.

The second deals quantitatively with the various specific quantities used in the study of radiation from the energy point of view, while the third deals with the ordinary photometric units and nomenclature.

The fourth chapter discusses the relation between the luminous and energy measurements of visible radiation.

A very clear treatment of the polarisation of light is given in the fifth chapter, and the theory of a number of different forms of analysers and polarisers is given, in which limitations are given more prominence than is usual in the treatment of the subject.

Chapter six deals with the reflection, absorption, transmission, and diffusion of light.

Chapter seven deals with the energetics of black body or complete radiation, while the final chapter briefly discusses colorimetry (which is however the subject of a later section of the encyclopaedia).

Throughout the work is extremely elegant and clear in style and up to the high standard we are accustomed to in Prof. Fabry's work.

H. B.

Einführung in die theoretische Optik. By Professor MAX PLANCK. Pp. vii + 184; 24 Figs. (Leipzig: S. Hirzel, 1927.) Price R.M. 6, bound R.M. 7.50.

This book forms the fourth volume of the author's *Einführung in die theoretische Physik*. It is divided into three sections; the first deals with the optics of isotropic homogeneous substances, the second with crystal optics, and the third with dispersion in isotropic substances. The treatment, which is very clear but necessarily condensed, is based on Maxwell's electromagnetic equations. For a beginner to understand all the steps

in the reasoning it is necessary to refer to several of the equations developed in the three previous volumes which deal respectively with mechanics, elasticity, and electricity and magnetism. In the last part of the book the author touches on the application of quantum mechanics to the problem of geometrical optics in non-homogeneous substances. The book can be recommended to those who wish to obtain a general idea of the theoretical treatment of optical problems but do not have time or ability to follow more detailed mathematical treatises on the subject.

Medical Research Council and Department of Scientific and Industrial Research. The Effect of Different Systems of Lighting on Output and Accuracy in Fine Work (Typesetting by Hand). Joint Report of the Industrial Fatigue Research Board and the Illumination Research Committee. By H. C. WESTON, M.J.Inst.E., and A. K. TAYLOR, A.C.G.I., M.I.E.E., A.M.I.C.E. Pp. iv + 12; 2 Figs. (London: H.M. Stationery Office, 1928.) Price 4*d*. net.

In a previous investigation it was found that, for maximum output with minimum errors, illumination of the order of 20 foot-candles was needed. In this enquiry the efficiencies attained with different lighting systems (direct, semi-indirect, semi-indirect plus local direct, indirect with partial daylight correction), all at 10 foot-candles, were determined. The effect of glare was also considered. Recommendations for the lighting of composing-rooms and shops where fine work is carried out are made.

T. S.

ABSTRACTS OF PATENT SPECIFICATIONS

270571. Brackensey, A. H., and others. REFLECTORS.

A reflex signal consists of a bi-convex lens of considerable thickness backed by a curved reflector. The curvature of the front surface of the lens is greater than that of the back surface, and that of the reflector has an intermediate value.

270659*. Hugershoff, R. STEREOGRAMS.

The movements of the measuring device are resolved into two components, one being applied to move the image carriers in one plane, while the other moves the reflectors of the measuring telescope in a plane at right angles to the first.

270683*. Spierer, C. MICROSCOPES.

In an illuminating device for the ultra-microscope, a central reflecting spot is provided on the front face of the second lens of the objective. The central portion of the field of light from the condenser, after passing the object, is thus reflected back upon it.

270790. Stimson, J. C. REFLEX SIGNALS.

In a reflector comprising a number of triangular or cubical reflecting units, the angles between the faces are somewhat distorted to produce a slightly spreading beam. The degree of polish may also be controlled with the same end in view. The reflector may be pressed from a single sheet.

270791. Beard, P. W. PROJECTION LAMPS.

Different arrangements involving the combination of a bi-concave and two plano-convex lenses of different aperture are utilised to produce either a broad or a narrow beam of light, free from the appearance of any image of the actual source.

R. J. T.

* Specification published before acceptance.